The second support weight distribution of the Kasami codes

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Abstract—We compute the second support weight distribution of the Kasami codes.

Index Terms—Kasami code, support weight distribution

The support weight distribution (SWD) of linear codes was introduced by Helleseth, Kløve, and Mykkeltveit [1]. From the SWD of a single code, they were able to determine the weight distribution of a corresponding infinite class of codes. After the introduction of the related weight hierarchy in [2], this problem received renewed interest, and in recent years, the SWD-s of particular codes [3], [4] and dual codes [5], [6], [7] have been studied. In this paper we give a short and simple calculation of the second SWD of the Kasami codes.

I. PRELIMINARIES

Let $GF(q)$ be the finite field of $q$ elements and $GF(q)^n$ a vector space of dimension $n$ with a fixed coordinate basis. An $[n,k]$ code $C$ over $GF(q)$ is a $k$-dimensional subspace of $GF(q)^n$. For any vector $x \in GF(q)$, the support $\chi(x)$ is defined as the set of coordinate positions where $x$ is non-zero. For a subset $S \subseteq GF(q)^n$, the support $\chi(S)$ is the union of supports of the members of $S$. The weight $w(x)$ or $w(S)$ of an element or a set is the cardinality of its support.

The weight hierarchy of a code $C$ is the sequence $(d_1, \ldots, d_k)$, where $d_r$ is the smallest weight of any $r$-dimensional subcode of $C$. The support weight distribution of $C$ is the array of parameters $A^r_i$ where $0 \leq i \leq n$ and $0 \leq r \leq k$, defined as the number of $r$-dimensional subcodes of $C$ with weight $i$.

Let $T_m$ denote the Frobenius trace from $GF(q^n)$ to $GF(q)$, defined as

$$T_m(x) = \sum_{i=0}^{m-1} x^{q^i}.$$ 

It is well known that

$$T_m(x + y) = T_m(x) + T_m(y),$$

$$T_m(x) = T_m(x^q),$$

and if $x$ runs through $GF(q^m)$, then $T_m(x)$ takes each value in $GF(q)$ exactly $q^{m-1}$ times. The original Kasami code is a binary code, so throughout the paper, we let $q = 2$ and write $Q = 2^m$. Thus $T_m : GF(Q) \rightarrow GF(2)$ and $T_{2m} : GF(Q^2) \rightarrow GF(2)$.

Definition 1 (The Kasami Codes) The Kasami code with parameters $[2^{2m-1} - 1, 3m, 2^{2m-1} - 2^{m-1}]$ is the set

$$K_m = \{ c(a, b) : a \in GF(Q^2), b \in GF(Q) \},$$

where

$$c(a, b) = (T_{2m}(ax) + T_m(bx^{Q+1}) : x \in GF(Q^2)^*).$$

The Kasami codes have three different non-zero weights, given by the following lemma.

Lemma 1 ([8]) The weight of a codeword $c(a, b) \in K_m$ is given by

$$w(c(a, b)) = \begin{cases} d_1 := 2^{2m-1} - 2^{m-1}, & \text{if } b \neq 0 \text{ and } T_m(a^{Q+1}/b) = 1, \\ m_1 := 2^{2m-1} + 2^{m-1}, & \text{if } b \neq 0 \text{ and } T_m(a^{Q+1}/b) = 0, \\ w_1 := 2^{2m-1}, & \text{if } b = 0 \text{ and } a \neq 0, \\ 0, & \text{if } b = 0 \text{ and } a = 0. \end{cases}$$

Remark 1 Given a non-zero $b \in GF(Q)$, there are $2^{m-1}$ choices for $a^{Q+1}$ giving $w(c(a, b)) = d_1$ and as many for $m_1$. For each non-zero value of $a^{Q+1}$, there are $(2^{m-1} - 1)/(2^m - 1) = 2^m + 1$ choices for $a$. Hence the number of codewords $c(a, b)$ for $b$ fixed of minimum and maximum weight are determined by

$$\#\{a : T_m(a^{Q+1}) = 1\} = (2^m + 1) \cdot 2^m - 1,$$

$$\#\{a : T_m(a^{Q+1}) = 0\} = (2^m + 1) \cdot (2^{m-1} - 1) + 1 = 2^m(2^m - 1).$$

The weight hierarchy of $K_m$ was studied in [8], and we will need several lemmata therefrom. Let $\gamma \in GF(Q^2)$, and define $V_\gamma := \{ c(\gamma b, b^2) : b \in GF(Q) \}$. Observe that $V_\gamma$ is a subcode of dimension $m$.

Lemma 2 ([8]) All the non-zero words of $V_\gamma$ have the same weight, which is $d_1$ if $T_m(\gamma^{Q+1}) = 1$ and $m_1$ if $T_m(\gamma^{Q+1}) = 0$.

Define

$$f(\gamma, a) := \gamma^{2Q}a^2 + \gamma^2a^{2Q} + a^{Q+1}.$$
and writing $z = a^{Q-1}$, we get that

$$\gamma^2 z^2 + z + \gamma^{2Q} = 0, \quad z^{Q+1} = 1. \quad (1)$$

Setting $u = \gamma^2 z$ and multiplying by $\gamma^2$, we get that (1) is equivalent to

$$u^2 + u + \gamma^{2(Q+1)} = 0, \quad u^{Q+1} = \gamma^{2(Q+1)}. \quad (2)$$

Set $b = \gamma^{2(Q+1)}$. From (2), we get that $u^2 = u + b$, which is, by repeated squaring and multiplication by $u$, equivalent to

$$u^{2m+1} = u^2 + T_m(b)u,$$

which is equal to $\gamma^{2(Q+1)}$ if and only if $T_m(b) = 1$ by Lemma 2. This proves the lemma.

II. SECOND SUPPORT WEIGHT DISTRIBUTION

Consider the two-dimensional subcodes of $K_m$. There are essentially eight types of such subcodes, which we denote by the weights of the three non-zero words as follows: $w.w.w, w.d.d, w.d.m, w.m.m, d.d.d, d.d.m, d.m.m,$ and $m.m.m$. Let $B_{x,x}$ denote the number of subcodes of Type $x.x.x$, and let $A^2_i$ be the number of two-dimensional subcodes of weight $i$. We distinguish four different cases. Let $B'_{x,x}$ denote the number of subcodes of Type $x.x.x$ resulting from Case $y$.

Let $D = (a, b)$ be a two-dimensional subcode, where $a = c(a_1, b_1)$ and $b = c(a_2, b_2)$ and $a + b = c(a_3, b_3)$. Recall that $a_3 = a_1 + a_2$ and $b_3 = b_1 + b_2$.

**Case 1** $b_1 = b_2 = b_3 = 0$.

The words of weight $2^{m-1}$ are $c(a, 0)$ where $a \neq 0$. So if $D$ has three words of weight $2^{m-1}$ it must be one of the $(2^{2m-1} - 1)/3$ 2-dimensional subcodes contained in

$$\{c(a, 0) : a \in GF(2^m)\}.$$

$$B^4_{w.w.w} = (2^{m-1})(2^{2m-1} - 1)/3.$$

**Case 2** $b_1 = b_2 \neq 0, b_3 = 0$.

There are $2^{m-1}$ choices for $b_1$. We have three possibilities, (1) $w(a) = w(b) = d_1$, (2) $w(a) = w(b) = m_1$, and (3) $w(a) = d_1$ whereas $w(b) = m_1$. For (1) and (2), $a$ and $b$ may be interchanged, so each possibility is counted twice. The number of $a$ values giving each weight is found by Remark 1.

$$B^2_{w.d.d} = 2^{m-2}(2^m - 1)(2^m + 1)((2^m + 1)2^{m-1} - 1),$$
$$B^2_{w.d.m} = 2^{m-2}(2^{2m} - 1)(2^m - 1),$$
$$B^2_{w.m.m} = 2^{m-2}(2^m - 1)^2(2^m + 1)(2^{m-1} - 1).$$

**Cases 3-4** $b_1, b_2, b_3$ distinct.

Define $\gamma_i = a_i/\sqrt{b_i}$. Observe that $\sqrt{b_3} = \sqrt{b_1} + \sqrt{b_2}$, because $(x + y)^2 = x^2 + y^2$ in characteristic 2. It follows that if $\gamma_1 = \gamma_2$, then

$$a_3 = \gamma_3 \sqrt{b_3} = \gamma_1 (\sqrt{b_1} + \sqrt{b_2}) = \gamma_1 \sqrt{b_3},$$

so $\gamma_3 = \gamma_1$ as well.

**Case 3** $\gamma_1 = \gamma_2 = \gamma_3$.

In this case, $D \subseteq V_3$. So either $D$ has three words of weight $m_1$ or three of weight $d_1$. There are $(2^m - 1)(2^{m-1} - 1)/3$ possible two-dimensional subcodes for each choice of $\gamma_1$; and the number of $\gamma_1$ values for each weight is found in Remark 1.

$$B^3_{d.d.d} = \frac{(2^m - 1)(2^{m-1} - 1)}{3} 2^{m-1}(2^m + 1),$$
$$B^3_{m.m.m} = \frac{(2^m - 1)(2^{m-1} - 1)}{3} 2^{m-1}(2^m - 1).$$

**Case 4** Distinct $\gamma_1, \gamma_2, \gamma_3$.

In this case, there is an $a' \in GF(Q^2)\setminus\{0\}$ such that

$$c(a_1, b_1) \in V_3 + c(a', 0),$$
$$c(a_2, b_2) \in V_3 + c(a', 0),$$
$$c(a_3, b_3) \in V_3.$$

The subcode $D$ is chosen by the following procedure.

1) Choose $\gamma_3$. There are $2^{2m}$ possibilities.

2) Choose $a' \neq 0$. There are $2^{2m} - 1$ possibilities.

3) Choose an unordered pair of points $b_1, b_2 \in GF(Q^2)$, which defines uniquely a pair of distinct points in $V_3 + c(a', 0)$. There are $(2^m - 1)(2^{m-1} - 1)$ possibilities.

Consider the case where $T_m(\gamma^{2(Q+1)}) = 0$, which implies that $w(c(a_3, b_3)) = m_1$. By Remark 1, there are $(2^m - 1)(2^{m-1})$ appropriate choices of $\gamma_3$. By Lemmata 3 and 4, for any $a' \neq 0, V_3 + c(a', 0)$ has $2^{2m-1} - 1$ words of weight $d_1$ and $2^{m-1} - 1$ words of weight $m_1$. Thus there are $2^{m-2} - 2^{m-1}$ pairs $(b_1, b_2)$ giving one word of weight $d_1$ and one of weight $m_1$. There are $(2^m - 1)(2^{m-2} - 1)$ pairs where both words have weight $d_1$, and $2^{m-2} - 2^{m-1}$ where both have weight $m_1$. Each subcode has been counted once for each maximum weight word it contains, since any such word may be $c(a_3, b_3)$. Thus we get

$$B^4_{d.d.m} = 2^{m-1}(2^m - 1)(2^{2m} - 1)(2^{m-1} - 1)/3,$$
$$B^4_{m.m.m} = 2^{m-1}(2^m - 1)(2^{2m} - 1)(2^{m-1} - 1)/2,$$
$$B^4_{m.m.m} = 2^{m-1}(2^m - 1)(2^{2m} - 1)(2^{m-1} - 1)/3.$$

Using Remark 4, $T$ is the number of words for Case 4, i.e.

$$T = (2^{4m} - 2^{2m})(2^{m-1} - 1)/3.$$
Theorem 1 The second support weight distribution of the $[2^{2m} - 1, 3m, 2^{2m-1} - 2^{m-1}]$ Kasami code is given by the expressions in Table I.

We have verified the 2nd SWD for some small Kasami codes by computer, and these numbers are shown in Table II.

It appears to be more difficult to determine higher order support weight distributions completely. The most difficult case is probably when all the $\gamma_i$ are distinct. For instance, studying a three-dimensional subcode, we have one non-zero word in $V_{\gamma_1}$ and two words in each of three cosets $V_{\gamma_1} + c(a_1, 0)$, $V_{\gamma_1} + c(a_2, 0)$, and $V_{\gamma_1} + c(a_1 + a_2, 0)$. Since only three out of the six coset points may be chosen freely, it is not obvious how to divine the weights of the remaining three. Maybe it can be done in combination with other methods.

REFERENCES


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His research interest include codes for digital fingerprinting and higher weights of linear codes.
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**Table I**

The second SWD of the Kasami codes.
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**TABLE II**

The 2nd SWD for some small Kasami codes.