On Weight Hierarchies of Product Codes
The Wei-Yang Conjecture and more

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Abstract

The weight of a code is the number of coordinate positions where no codeword is zero. The $r$th minimum weight $d_r$ is the least weight of an $r$-dimensional subcode. Wei and Yang conjectured a formula for the minimum weights of some product codes. The conjecture is proved in two different ways, each with interesting side-results.

Key words: product code, projective multiset, weight hierarchy, Segre embedding, chain condition

1 Introduction


In the special case where both the component codes satisfy the chain condition they found an upper bound on the weight hierarchy. They conjectured that this bound is always satisfied with equality.
Two different proofs of the conjecture have appeared recently [11,9]. Each of the proofs give interesting generalisations of the Wei-Yang conjecture. In the sequel we will survey the results from both papers, as well as related results from the last decade. A new result appears in Theorem 8. Finally we suggest some open problems for further study.

2 Product Codes

An \([n, k]\) code is a \(k\)-dimensional subspace \(C \subseteq V\) of some \(n\)-dimensional vector space \(V\). The support of a vector \(c = (c_1, c_2, \ldots, c_n) \in V\) is the set
\[
\chi(c) := \{i \mid c_i \neq 0\},
\]
and the support of a subset \(S \subseteq V\) is the set
\[
\chi(S) := \bigcup_{c \in S} \chi(c).
\]
The weight hierarchy of the code \(C \subseteq V\) is the sequence
\[
(d_1(C), d_2(C), \ldots, d_k(C)),
\]
where
\[
d_r(C) := \min\{\#\chi(D) \mid D \subseteq C, \dim D = r\}.
\]
Clearly \(d_1(C)\) is the minimum distance, and for convenience we put \(d_0(C) = 0\).

The chain condition says that there exists a sequence of subcodes
\[
\{0\} = D_0 < D_1 < \ldots < D_k = C,
\]
such that \(D_i\) has dimension \(i\) and weight \(d_i(C)\). Codes satisfying the chain condition are called chained codes.

**Remark 1** Almost all codes violate the chain condition [4]. However, a lot of interesting codes do satisfy the chain condition, such as the Hamming, Reed-Muller, MDS and extended Golay codes. In particular, codes with high generalised Hamming weights tend to satisfy the chain condition.

Let \(C_1\) be an \([n_1, k_1]\) code and \(C_2\) an \([n_2, k_2]\) code over the same field \(F\). The product code \(C_1 \otimes C_2\) is the tensor product of \(C_1\) and \(C_2\) as vector spaces over \(F\). In other words
\[
C_1 \otimes C_2 = \langle a \otimes b \mid a \in C_1, b \in C_2 \rangle,
\]
where
\[
  a \otimes b = (a_i b_j \mid 1 \leq i \leq n_1, 1 \leq j \leq n_2),
\]
\[
  a = (a_1, a_2, \ldots, a_{n_1}),
\]
\[
  b = (b_1, b_2, \ldots, b_{n_2}).
\]

The product code is an \([n_1 n_2, k_1 k_2]\) code.

A \((k_1, k_2)\)-partition of \(r\) is a non-increasing sequence \(\pi = (t_1, t_2, \ldots, t_{k_1})\) such that \(t_1 + t_2 + \ldots + t_{k_1} = r\) and \(t_i \leq k_2\). Let \(\mathcal{P}(k_1, k_2; r)\) denote the set of all \((k_1, k_2)\)-partitions of \(r\).

We define
\[
  \nabla(\pi) := \sum_{i=1}^{k_1} (d_i(C_1) - d_{i-1}(C_1))d_{t_i}(C_2), \quad \pi \in \mathcal{P}(k_1, k_2; r),
\]
\[
  d_r^*(C_1 \otimes C_2) := \min \{ \nabla(\pi) \mid \pi \in \mathcal{P}(k_1, k_2; r) \}.
\]

The number \(d_r^*\) was first defined in [15], and the following theorem was proved.

**Theorem 2 (Wei–Yang 1993)** If \(C_1\) and \(C_2\) are chained codes, then \(d_r(C_1 \otimes C_2) \leq d_r^*(C_1 \otimes C_2)\) for all \(r\).

The Wei-Yang conjecture says that \(d_r(C_1 \otimes C_2) = d_r^*(C_1 \otimes C_2)\) for chained codes \(C_1\) and \(C_2\). This has been proved for \(r \leq 4\) by Barbero and Tena [1]. Following the conjecture, \(d_r^*\) has been determined for some linear codes by Helleseth and Klove in [7]. It has also been proved that \(d_r^*(C_1 \otimes C_2) = d_r^*(C_2 \otimes C_1)\). Jeng Yune Park [10] has computed \(d_r^*\) for products of certain classes of codes and shown that the conjecture holds for these classes.

**Theorem 3 (Schaathun 2000)** If \(C_1\) and \(C_2\) are arbitrary linear codes, then \(d_r(C_1 \otimes C_2) \geq d_r^*(C_1 \otimes C_2)\) for all \(r\).

The theorem has been proved in the language of projective multiset. Obviously Theorem 2 and 3 together prove the Wei-Yang conjecture.

A projective multiset is a collection of projective points which are not necessarily distinct. It is often presented as a map
\[
  \gamma : \text{PG}(k - 1, q) \to \{0, 1, 2, \ldots\}
\]
where the value \(\gamma(x)\) of a point \(x\) is the number of times \(x\) occurs in the collection. We also define the value of a subset \(S \subseteq \text{PG}(k - 1, q)\) to be \(\gamma(S) = \sum_{x \in S} \gamma(x)\).
There is a well-known one-to-one correspondence between equivalence classes of projective multisets on \( \text{PG}(k-1,q) \) and equivalence classes of linear \( q \)-ary codes of dimension \( k \). See for instance [13,5,11]. The projective multiset \( \gamma \) corresponding to some \([n,k]\) code \( C \) is obtained by taking the columns of some generator matrix for \( C \) as projective points.

The largest value of any subspace of codimension \( r \) in \( \text{PG}(k-1,q) \) is \( n-d_r(C) \). Hence the weight hierarchy can be computed in the language of projective multisets.

When \( a \) and \( b \) are projective points, the map \( a,b \mapsto a \otimes b \) is known as the Segre embedding. The image under the Segre embedding is called the Segre variety. The proof of Theorem 3 is based on the following key fact.

**Proposition 4** Let \( \gamma_A, \gamma_B, \) and \( \gamma_C \) be the projective multisets corresponding to linear codes \( A, B, \) and \( C = A \otimes B \), respectively. We have:

\[
\gamma_C(a \otimes b) = \gamma_A(a) \cdot \gamma_B(b), \quad \forall a \in \mathbb{P}^{k_A-1}, \forall b \in \mathbb{P}^{k_B-1}, \forall c \notin Y,
\]

where \( Y \) is the Segre variety.

**Theorem 5** For any two linear codes \( C_1 \) and \( C_2 \), we have

\[
d^*_r(C_1 \otimes C_2) = d_r(C_1 \otimes C_2), \quad r \in \{0, 1, 2, k-2, k-1, k\}.
\]

This is well known for \( r = 0, 1, k \). Wei and Yang proved it for \( r = 2 \). The proof for \( r = k-2, k-1 \) appears in [11], based on the language of projective multisets. For a six-dimensional product code \( C_1 \otimes C_2 \), we may or may not have \( d^*_3(C_1 \otimes C_2) = d_3(C_1 \otimes C_2) \), see [11].

3 General Product Codes

Generalising the results from Section 2 we are lead to the following question:

*Given a family \( C_1, C_2, \ldots, C_t \) of linear codes, what can we say about the weight hierarchy of \( C_1 \otimes C_2 \otimes \ldots \otimes C_t \)?*

It is worth noting that the affirmative answer to the Wei-Yang conjecture has no implications for general product codes. Even if \( C_1 \) and \( C_2 \) are chained, \( C_1 \otimes C_2 \) may be non-chained.

To answer the question, we generalise the definition of \((k_1,k_2)\)-partitions to \((k_1,k_2,\ldots,k_t)\)-partitions.
Define
\[ \mathcal{M}_t := \{ i = (i_1, i_2, \ldots, i_{t-1}) \mid 1 \leq i_j \leq k_j, 1 \leq j < t \}. \]
Note that \( \mathcal{M}_2 \) is just the set of integers from 1 through \( k_1 \). Hence an element \( \pi \in \mathcal{P}(k_1, k_2; r) \) is a map \( \pi : \mathcal{M}_2 \to \{0, 1, \ldots, k_2\} \) where \( \pi(i) = t_i \).

**Definition 6** Let \( \pi \) be a map \( \mathcal{M}_t \to \{0, 1, \ldots, k_t\} \) given by \( i \mapsto t_i \). We call \( \pi \) a \((k_1, k_2, \ldots, k_t)\)-partition of \( r \) if

1. \( \sum_{i \in \mathcal{M}_t} t_i = r \).
2. \( \pi \) is a decreasing function in each coordinate, i.e.
   \[ t_{i_1, \ldots, i_j, \ldots, i_{t-1}} \leq t_{i_1, \ldots, i_{j-1}, \ldots, i_{t-1}} \]
   for \( j = 1, \ldots, t-1 \) and \( 1 < i_j \).

We define the general expression for \( d^*_r \) as follows:

\[ d^*_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) := \min \{ \nabla(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r) \} \]

where

\[ \nabla(\pi) = \sum_{i \in \mathcal{M}_t} \prod_{j=1}^{t-1} (d_{i_j}(C_j) - d_{i_{j-1}}(C_j))d_{\pi(i)}(C_t). \]

**Theorem 7 (Martínez-Pérez-Willems 2000)**
If \( C_1, C_2, \ldots, C_t \) are chain codes, then

\[ d_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) = d^*_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t). \]

The Wei-Yang conjecture is just the special case of the theorem with \( t = 2 \).

The original proof of Theorem 7 above uses only elementary facts from linear algebra. For each subspace \( D \subseteq C_1 \otimes C_2 \otimes \ldots \otimes C_t \) there is a so-called normal subspace \( \bar{D} \) such that \( \dim D = \dim \bar{D} \) and \( \#\chi(D) \leq \#\chi(\bar{D}) \). Such a normal subspace \( \bar{D} \) is naturally associated to a \((k_1, \ldots, k_t)\)-partition \( \pi \) and has support size \( \nabla(\pi) \).

The following theorem is a new result. The proof is based on projective multisets, and it can also be extended to prove Theorem 7. We hope to publish the proof elsewhere.

**Theorem 8** If \( C_1, C_2, \ldots, C_t \) are arbitrary linear codes, then

\[ d_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) \geq d^*_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t). \]
By Theorem 3, we can obtain a lower bound for general product codes by recursion, but the theorem above is stronger than that. There is an analogue of Theorem 8 [12] for the greedy weights, which was introduced in [2,3].

A  Some Future Work

The support weight distribution was introduced in [8]. It was proved that if the support weight distribution for $C$ is known, then one can also find the weight distribution of $C \otimes S$, where $S$ is a simplex code. The support weight distribution is only known for a very few classes of codes. Is it possible to find the support weight distribution of some classes of product codes, such as the product of two simplex codes?

David Forney [6] proved that the generalised Hamming weights give a lower bound on the state complexity of a minimal trellis. It was proved that this bound is met with equality with some optimal bit ordering if and only if the code meets the so-called two-way chain condition. Is it possible to determine completely the state complexity of a product code, given the state complexities of the component codes?

References


