Upper bounds on Weight Hierarchies of Extremal Non-Chain Codes

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Abstract

The weight hierarchy of a linear \([n, k; q]\) code \(C\) over \(\mathbb{GF}(q)\) is the sequence \((d_1, d_2, \ldots, d_k)\) where \(d_r\) is the smallest support weight of an \(r\)-dimensional subcode of \(C\). Linear codes may be classified according to a set of chain and non-chain conditions, the extreme cases being codes satisfying the chain condition (due to Wei and Yang) and extremal, non-chain codes (due to Chen and Kløve). This paper gives upper bounds on the weight hierarchies of the latter class of codes.

Key words: Weight hierarchy, chain condition, linear codes, projective multiset

1 Introduction

The concept of generalised Hamming weights was introduced as early as 1977 by Helleseth et al. [8] in their study of weight distributions of irreducible cyclic codes. The term ‘generalised Hamming weight’ was introduced by Wei in 1991 [14]. He used the parameters to analyse an application of codes on the Wire-Tap Channel of type II, which had been introduced in 1984 by Ozarow and Wyner [11]. During the nineties, several researchers have studied the generalised Hamming weights of linear codes.

The chain condition was introduced by Wei and Yang [15]. Chen and Kløve [2] introduced the opposite extreme, extremal non-chain codes. Known codes with high generalised Hamming weights tend to satisfy the chain condition. Cohen et al. [6] argue that some non-chain codes may have other advantages. Our interest is purely mathematical however.

Chen and Kløve found tight upper bounds for non-binary, four-dimensional, extremal non-chain codes [2]. Later they have also found all possible weight hierarchies of four-dimensional binary codes [5]. In this paper we generalise...
their upper bounds to arbitrary dimension, and these bounds are the best possible in dimension 5 and lower.

1.1 Notation and definitions

Throughout this paper $C$ will denote an $[n, k + 1; q]$ code, i.e. a linear code of length $n$ and dimension $k + 1$ over the Galois field $GF(q)$ with $q$ elements. Codes of dimension $k + 1$ will be studied in a projective space $PG(k, q)$ of dimension $k$ and order $q$.

Given a code $C$ we define the support $\chi(C)$ to be the set of positions where not all codewords of $C$ are zero, i.e.

$$\chi(C) := \{i \mid \exists (x_1, x_2, \ldots, x_n) \in C, \text{ s.t. } x_i \neq 0\}.$$ 

The support weight of $C$ is the size of $\chi(C)$, and we denote it $w_S(C)$, i.e.

$$w_S(C) := \#\chi(C).$$

For $0 \leq r \leq k + 1$, the $r$th generalised Hamming weight $d_r$ of $C$ is the least support weight of an $r$-dimensional subcode of $C$. The sequence $(d_1, d_2, \ldots, d_{k+1})$ is called the weight hierarchy of $C$. The minimum weight of the code is $d = d_1$.

We note that by adding a zero-position to $C$, we get an $[n + 1, k + 1; q]$ code with the same weight hierarchy as $C$. Without loss of generality, we can restrict our study to codes without zero-positions. In other words, we assume that $d_{k+1} = n$.

Two linear codes are equivalent if one can be obtained from the other by permuting coordinate positions or by multiplying some coordinate by a non-zero scalar. We note that equivalent codes have the same weight hierarchy.

1.2 Codes in projective geometry

We let $G$ denote a generator matrix of $C$. The value (or multiplicity) $\nu(x)$ of $x \in GF(q)^{k+1}$ is the number of occurrences of $x$ as a column in $G$. Replacing some column $x$ with $ax$ for some non-zero scalar $a$ leads to an equivalent code. Thus we can consider the columns of $G$ to be projective points, and an equivalence class of codes is uniquely determined by giving the map

$$\nu : PG(k, q) \to \mathbb{N}_0 := \{0, 1, \ldots\}.$$
This concept has been studied by several authors using different terminology. Dodunekov and Simonis [7] give an historic overview, and they prefer to call \( \nu \) a projective multiset. In this paper we prefer to call it a \textit{value assignment}, as did Chen and Kløve [2]. Tsfasman and Vladut [13] studied an equivalent concept called a projective system.

An arbitrary map \( \nu : \text{PG}(k, q) \rightarrow \mathbb{N}_0 \) is called a value assignment even if it is not defined from a code. We call it \textit{non-degenerate} if there are \( k + 1 \) projectively independent points \( p_0, p_1, \ldots, p_k \) such that \( \nu(p_i) \geq 1 \) for all \( i \). By taking \( \nu(x) \) not necessarily distinct representatives for each projective point and taking an ordering on all these representatives, we get a matrix \( G \). This matrix \( G \) is a generator matrix of a code if and only if its rank is \( k + 1 \), that is if \( \nu \) is non-degenerate.

We define the value of a set of points as follows

\[
\nu(U) := \sum_{x \in U} \nu(x), \quad \forall U \subseteq \text{PG}(k, q).
\]

Let \( \text{PG}^{(m)}(k, q) \) be the set of \textit{m-spaces} or \textit{m}-dimensional subspaces of \( \text{PG}(k, q) \). Note that \( \text{PG}^{(0)}(k, q) \) is the collection of subsets of cardinality 1; both \( P \in \text{PG}(k, q) \) and \( \{P\} \in \text{PG}^{(0)}(k, q) \) will be called a point. The 1-, 2- and \((k-1)\)-spaces are called lines, planes, and hyperplanes, respectively. The only \((-1)\)-space is the empty set.

The join of \( \Pi_r \) and \( \Pi_s \), denoted \( \Pi_r \cup \Pi_s \), is the intersection of all subspaces containing the union \( \Pi_r \cup \Pi_s \). If \( p_0, p_1, \ldots, p_m \in \text{PG}(k, q) \) are projectively independent points, we write \( \langle p_0, p_1, \ldots, p_m \rangle \) for their join. We define the following shorthand notation,

\[
\theta(n) := \frac{q^{n+1} - 1}{q - 1} = \sum_{i=0}^{n} q^i,
\]

and recall that \( \theta(k) \) is the cardinality of \( \text{PG}(k, q) \).

### 1.3 Subcodes and the value assignments

From now on we let \( \nu : \text{PG}(k, q) \rightarrow \mathbb{N}_0 \) be the value assignment corresponding to \( \mathcal{C} \). There is a one-to-one correspondence between subcodes of \( \mathcal{C} \) of dimension \( r \) and subspaces of \( \text{PG}(k, q) \) of dimension \( k - r \). We write \( D^* \) for the projective subspace corresponding to a subcode \( D \subseteq \mathcal{C} \), and \( \Pi^* \) for the subcode corresponding to \( \Pi \subseteq \text{PG}(k, q) \). If \( D_1 \subseteq D_2 \), then \( D_1^* \supseteq D_2^* \). It is known [9,13] that \( d_{k+1} - w_s(D) = \nu(D^*) \).

We define the weight hierarchy \((d_1, \ldots, d_{k+1})\) of a value assignment \( \nu \) by letting
Let $n - d_r$ be the greatest value of a subspace of codimension $r$ in $\text{PG}(k, q)$. Obviously the correspondence between value assignments and codes preserves the weight hierarchy. Note that a value assignment is non-degenerate if and only if $d_1 > 0$. All value assignments encountered in this paper are non-degenerate.

The difference sequence $(\delta_0, \delta_1, \ldots, \delta_k)$ of a code or of a value assignment is defined by

$$\delta_j := d_{k+1-j} - d_{k-j}, \quad j = 0, 1, \ldots, k.$$  

We note that the difference sequence is easily computed from the weight hierarchy and vice versa. We say that the difference sequence $(\delta_0, \delta_1, \ldots, \delta_k)$ has dimension $k + 1$. The elements of the difference sequence of a code or non-degenerate value assignment are positive, due to the strict monotonicity of the generalised Hamming weights.

The existence of a linear code with weight hierarchy $(d_1, d_2, \ldots, d_{k+1})$ is equivalent to the existence of a non-degenerate value assignment $\nu$ such that,

$$\max\{\nu(\Pi_m) \mid \Pi_m \in \text{PG}^{(m)}(k, q)\} = \sum_{i=0}^{m} \delta_i, \quad -1 \leq m \leq k.$$  

The set of $m$-spaces of maximum value is denoted by $M_m$,

$$M_m(\nu) := \left\{ \Pi_m \mid \Pi_m \in \text{PG}^{(m)}(k, q) \land \nu(\Pi_m) = \sum_{i=0}^{m} \delta_i \right\}, \quad -1 \leq m \leq k.$$  

When no ambiguity is expected, we write $M_m = M_m(\nu)$.

Given an $m$-space $\Pi_m \in \text{PG}^{(m)}(k, q)$, we can restrict the value assignment $\nu$ to this subspace and study

$$\nu' = \nu|_{\Pi_m} : \Pi_m \to \mathbb{N}_0.$$  

If $\Pi_m \in M_m(\nu)$, the monotonicity of the weight hierarchy ensures that any proper subspace of $\Pi_m$ has lower value. In this case $\nu'$ is non-degenerate, and thus defines a code $D$, which is actually the code obtained by puncturing $C$ on each coordinate in $\chi(\Pi_m)$. We write $M_i(\Pi_m) M_i(\nu|_{\Pi_m})$ for $-1 \leq i \leq m$.

### 1.4 The Chain Condition

The chain condition was introduced by Wei and Yang [15], and it says that

$$\forall i \text{ s.t. } 0 \leq i \leq k - 1 \quad \exists \Pi_i \in M_i \quad \text{s.t. } \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{k-1}.$$  

We will refer to codes satisfying this condition as chained codes.
We define a number of subconditions, which are implications of the chain condition. For all \( i \) and \( j \) such that \( 0 \leq i < j \leq k - 1 \), we have the condition,

\[(C_{i,j}) : \exists \Pi_i \in M_i \exists \Pi_j \in M_j \text{ s.t. } \Pi_i \subset \Pi_j.\]

The negations of these conditions, \((N_{i,j}) := \neg(C_{i,j})\), will be called non-chain conditions.

Analogous to the definition by Chen and Klove [2], we define extremal non-chain codes of arbitrary dimension to be codes that satisfy all of the non-chain conditions \((N_{i,j})\). The difference sequence of an extremal non-chain code will be called an ENDS (extremal non-chain difference sequence).

2 Upper bounds

2.1 The general upper bound

**Theorem 1** If \((\delta_0, \delta_1, \ldots, \delta_k)\) is an ENDS and \(1 \leq m \leq k - 1\), then

\[\delta_m \leq q^m \delta_0 - \sum_{i=0}^{m} q^i.\]

If this bound holds with equality for \(m = \bar{m} > 1\), then it also holds with equality for \(m = \bar{m} - 1\).

The proof of this theorem is quite tedious, and we have to start with some auxiliary results.

**Definition 2** We say that an ENDS is \(m\)-optimal, \(1 \leq m \leq k - 1\), if it satisfies the bound on \(\delta_m\) from Theorem 1 with equality. An extremal non-chain code \(C\) is \(m\)-optimal if its difference sequence is an \(m\)-optimal ENDS.

**Lemma 3** Given an arbitrary code with difference sequence \((\delta_0, \delta_1, \ldots, \delta_k)\), we have \(\delta_k \leq q\delta_{k-1}\).

**PROOF.** Take some \(\Pi_{k-2} \in M_{k-2}\). There are \(q + 1\) \((k - 1)\)-spaces through \(\Pi_{k-2}\), and for every such subspace \(\Pi_{k-1}\) we have

\[\nu(\Pi_{k-1} \setminus \Pi_{k-2}) \leq \delta_{k-1}.\]

The geometry is partitioned into \(q + 1\) disjoint subsets of the form \(\Pi_{k-1} \setminus \Pi_{k-2}\),
beside \( \Pi_{k-2} \). Hence
\[
\sum_{i=0}^{k} \delta_i \leq (q + 1)\delta_{k-1} + \sum_{i=0}^{k-2} \delta_i.
\]
The lemma follows immediately. \( \Box \)

**Lemma 4** Let \((\delta_0, \delta_1, \ldots, \delta_k)\) be the difference sequence of some non-degenerate value assignment \(\nu\), and \((\delta'_0, \delta'_1, \ldots, \delta'_{k-1})\) the difference sequence of \(\nu|_{\Pi_{k-1}}\) for some \(\Pi_{k-1} \in M_{k-1}\). Then \(\delta_{k-1} \leq \delta'_{k-1}\).

**PROOF.** We have \(\Pi_{k-1} \in M_{k-1}(\Pi_{k-1}) \subseteq M_{k-1}(\nu)\). Let \(\Pi_{k-2} \in M_{k-2}(\nu)\) and \(\Pi_{k-2} \in M_{k-2}(\Pi_{k-1})\). Clearly \(\nu(\Pi_{k-2}) \leq \nu(\Pi_{k-2})\). Hence
\[
\delta_{k-1} = \nu(\Pi_{k-1}) - \nu(\Pi_{k-2}) \leq \nu(\Pi_{k-1}) - \nu(\Pi'_{k-2}) = \delta'_{k-1},
\]
as required. \( \Box \)

**Lemma 5** Let \(\nu\) be the value assignment of an extremal non-chain code with difference sequence \((\delta_0, \delta_1, \ldots, \delta_k)\). If \(\Pi_m \in M_m\) where \(0 \leq m \leq k\) and \(\nu|_{\Pi_m}\) has difference sequence \((\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m)\), then \(\delta_m \leq \varepsilon_m - 1\).

**PROOF.** This goes almost like the proof of Lemma 4, except that since the code is extremal non-chain, we get a stronger bound. We have \(\Pi_m \in M_m(\Pi_m) \subseteq M_m(\nu)\). Let \(\Pi_{m-1} \in M_{m-1}(\nu)\) and \(\Pi'_{m-1} \in M_{m-1}(\Pi_m)\). Since the code is extremal non-chain, we have \(\nu(\Pi'_m) < \nu(\Pi_m)\). Hence
\[
\delta_m = \nu(\Pi_m) - \nu(\Pi_{m-1}) \leq \nu(\Pi_m) - (\nu(\Pi'_{m-1}) + 1) = \varepsilon_m - 1,
\]
as required. \( \Box \)

**Lemma 6** If \(k \geq 2\) and \((\delta_0, \delta_1, \ldots, \delta_k)\) satisfies (N0.1), then \(\delta_1 \leq q\delta_0 - (q + 1)\) and \(\delta_0 \geq 2\).

**PROOF.** A line consists of \(q + 1\) points, and by (N0.1), \(\delta_1 + \delta_0 \leq (q + 1)(\delta_0 - 1)\). Hence \(\delta_1 \leq q\delta_0 - (q + 1)\). Also if \(\delta_0 \leq 1\), then \(\delta_1 \leq -1\), which is absurd. \( \Box \)

**Proof of Theorem 1.** The proof goes by induction on \(m\), so we assume that the theorem holds for every \(m < \bar{m}\). Lemma 6 proves it for \(m = 1\). Now we consider a code \(\mathcal{C}\) such that
\[
\delta_{\bar{m}} \geq q^\bar{m}\delta_0 - \theta(\bar{m}) \quad (1)
\]
\[
\delta_{m} \leq q^m\delta_0 - \theta(m), \quad \forall m \leq \bar{m} - 1. \quad (2)
\]
Our aim is to prove that then we must have equality both in (1) and in (2).

Take an arbitrary \( \Theta_\bar{m} \in M_\bar{m}(\mathcal{C}) \), and let

\[
\Theta_0 \subset \Theta_1 \subset \ldots \subset \Theta_{\bar{m}-1} \subset \Theta_\bar{m}
\]

be a chain such that \( \Theta_i \in M_i(\Theta_{i+1}) \) for \( 0 \leq i \leq \bar{m} - 1 \). Let \((\varepsilon_0^{(i)}, \ldots, \varepsilon_i^{(i)})\) be the difference sequence of \( \nu|_{\Theta_i} \).

By Lemma 5 and (1), we get

\[
\varepsilon_\bar{m}^{(\bar{m})} \geq \delta_\bar{m} + 1 \geq q^\bar{m} \delta_0 - \theta(\bar{m}) + 1. \tag{3}
\]

Lemma 4 and 3 give

\[
\varepsilon_{\bar{m}-1}^{(\bar{m}-1)} \geq \varepsilon_{\bar{m}-1}^{(\bar{m})} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q} \right\rceil. \tag{4}
\]

Repeating this argument \( \bar{m} \) times and substituting from (3), we obtain

\[
\varepsilon_0^{(0)} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q^{\bar{m}}} \right\rceil \geq \left\lceil \frac{q^\bar{m} \delta_0 - \theta(\bar{m}) + 1}{q^{\bar{m}}} \right\rceil = \delta_0 - 1.
\]

Clearly \( \varepsilon_0^{(0)} \) is the value of \( \Theta_0 \), which is a point in \( \Theta_{\bar{m}} \in M_\bar{m}(\mathcal{C}) \). Since \( \mathcal{C} \) is extremal non-chain, we have \( \varepsilon_0^{(0)} \leq \delta_0 - 1 \). We conclude that

\[
\varepsilon_0^{(l)} = \nu(\Theta_0) = \delta_0 - 1, \quad \forall l, \ 0 \leq l \leq \bar{m}. \tag{5}
\]

We assume for induction on \( j \) that for all \( j < i \) where \( 0 < i < \bar{m} \), we have

\[
\varepsilon_j^{(l)} = \varepsilon_j^{(i)} = q^l \delta_0 - \theta(j), \quad \forall l, \ \text{s.t.} \ j \leq l \leq \bar{m}. \tag{6}
\]

First we prove that it also holds for \( l = j = i \). Repeating the argument of (4) \( \bar{m} - i \) times, we get

\[
\varepsilon_i^{(i)} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q^{\bar{m}-i}} \right\rceil \geq \left\lceil \frac{q^\bar{m} \delta_0 - \theta(\bar{m}) + 1}{q^{\bar{m}-i}} \right\rceil = q^i \delta_0 - \theta(i). \tag{7}
\]

Now \( \varepsilon_i^{(i)} = \nu(\Theta_i) - \nu(\Theta_{i-1}) \). Since \( \mathcal{C} \) is extremal non-chain, we get by (2) that

\[
\nu(\Theta_i) \leq \sum_{j=0}^i \delta_j - 1 \leq \sum_{j=0}^i [q^j \delta_0 - \theta(j)],
\]
and according to the induction hypothesis (6), we have
\[ \nu(\Theta_{i-1}) = \sum_{j=0}^{i-1} \varepsilon_j^{(i-1)} = \sum_{j=0}^{i-1} \varepsilon_j = \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)]. \] (8)

Combining these expressions, we get an upper bound on \( \varepsilon_i^{(i)} \):
\[
\varepsilon_i^{(i)} = \nu(\Theta_i) - \nu(\Theta_{i-1}) \\
\leq \sum_{j=0}^{i} [q^j \delta_0 - \theta(j)] - \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)] = q^i \delta_0 - \theta(i). \] (9)

Combining the upper and lower bounds (7) and (9), we conclude by induction that
\[ \varepsilon_i^{(i)} = q^i \delta_0 - \theta(i), \quad i = 0, \ldots, \bar{m} - 1. \] (10)

From (8) and (2) we can see that
\[
\sum_{j=0}^{i-1} \delta_j - 1 \geq \nu(\Theta_{i-1}) = \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)] \geq \sum_{j=0}^{i-1} \delta_j - 1,
\]
Hence \( \delta_{i-1} = q^{i-1} \delta_0 - \theta(i - 1) \). Also
\[
\varepsilon_i^{(i)} + \nu(\Theta_{i-1}) = q^i \delta_0 - \theta(i) + \nu(\Theta_{i-1}) = \nu(\Theta_i) \leq \sum_{j=0}^{i} \delta_j - 1.
\]
Hence \( q^i \delta_0 - \theta(i) \leq \delta_i \), and combining with (2), we get \( \delta_i = q^i \delta_0 - \theta(i) \).

It follows from this argument that \( \Theta_i \in M_i(\Theta_l) \) and \( \Theta_{i-1} \in M_{i-1}(\Theta_l) \), for all \( l \) such that \( i \leq l \leq \bar{m} \), and hence \( \varepsilon_i^{(i)} = \varepsilon_i^{(i)} \). It follows by induction that \( \delta_i = q^i \delta_0 - \theta(i) \) for \( i = 1, 2, \ldots, \bar{m} - 1 \).

We have
\[
\varepsilon_{\bar{m}}^{(\bar{m})} = \nu(\Theta_{\bar{m}}) - \nu(\Theta_{\bar{m}-1}) = \sum_{i=0}^{\bar{m}} \delta_i - \left( \sum_{i=0}^{\bar{m}-1} \delta_i - 1 \right) = \delta_{\bar{m}} + 1,
\]
and by Lemmas 3 and 4 and (10), we get
\[
\delta_{\bar{m}} + 1 = \varepsilon_{\bar{m}}^{(\bar{m})} \leq q^{\bar{m}} \varepsilon_{\bar{m}-1} \leq q^\bar{m}^{(\bar{m}-1)} = q^\bar{m} \delta_0 - \theta(\bar{m}) + 1.
\]

Combining with the lower bound from (1) we get
\[
\delta_{\bar{m}} = \varepsilon_{\bar{m}}^{(\bar{m})} - 1 = q^\bar{m} \delta_0 - \theta(\bar{m}),
\]
and the theorem follows by induction. \( \square \)
Figure 1. Representation of $PG(4, 2)$ for the proof of Theorem 9. Black lines are in $\Pi_3$, dashed lines in $\Pi_2$, and dotted lines are in neither. White points are in $\Pi_1$. The point $L_1$ and $\Pi_1$ span $\mathcal{L}_1$, and $L_2$ and $\Pi_1$ span $\mathcal{L}_2$.

**Corollary 7** Let $C$ be an $m$-optimal code with difference sequence $(\delta_0, \delta_1, \ldots, \delta_k)$ for some $m$ such that $1 \leq m \leq k - 1$. For every $\Pi_m \in \mathcal{M}_m$, $\nu|_{\Pi_m}$ corresponds to a chained code with difference sequence $(\delta_0 - 1, \delta_1, \delta_2, \ldots, \delta_{m-1}, \delta_m + 1)$.

**Proof.** In the proof of Theorem 1 we proved that $\Theta_i \in \mathcal{M}_i(\Theta_m) = \mathcal{M}_i(\Pi_m)$, and we found the difference sequence as given in the corollary. □

**Remark 8** We know from Lemma 6 that $\delta_0 \geq 2$, so the difference sequence has only positive elements as expected. Writing

$$(\varepsilon_0 = \delta_0 - 1, \varepsilon_1 = \delta_1, \ldots, \varepsilon_{m-1} = \delta_{m-1}, \varepsilon_m = \delta_m + 1)$$

for the difference sequence of $\nu|_{\Pi_m}$, we have $\varepsilon_i = q\varepsilon_{i-1} - 1$ for $i = 1, \ldots, m - 1$ and $\varepsilon_m = q\varepsilon_{m-1}$.

2.2 Binary case

For binary codes we have a special bound, which also implies that binary codes cannot be $(k - 1)$-optimal if $k \geq 3$.

**Theorem 9** If $(\delta_0, \delta_1, \ldots, \delta_k)$, $k \geq 3$, is a binary ENDS, then

$$\delta_{k-1} \leq 2^{k-2}\delta_1 - 2 - 2^{k-2}.$$
**Proof.** Take $\Pi_{k-1} \in M_{k-1}$ and $\Pi_{k-2} \in M_{k-2}$, and let

$$
\Pi_{k-3}P \cap \Pi_{k-2}.
$$

Because the code is extremal non-chain, $\Pi_{k-3}$ is a $(k-3)$-space. Also let $\{P\} \in M_0$.

Define

$$
S := \Pi_{k-2}\backslash \Pi_{k-3} = \{S_i \mid i = 1, 2, \ldots, 2^{k-2}\},
$$

$$
\ell_i := \langle P, S_i \rangle = \{P, S_i, T_i\}, \quad i = 1, 2, \ldots, 2^{k-2}.
$$

Every line through $P$ meets $\Pi_{k-1}$, so the points $T_i$ are in $\Pi_{k-1}$. Define the set

$$
T := \{T_i \mid i = 1, 2, \ldots, 2^{k-2}\}.
$$

Because the code is an ENDS, $\nu(\ell_i) \leq \delta_0 + \delta_1 - 1$ for all $i$. Hence

$$
\nu(T_i) \leq \delta_1 - \nu(S_i) - 1, \quad i = 1, 2, \ldots, 2^{k-2},
$$

$$
\nu(T) \leq 2^{k-2}\delta_1 - \nu(S) - 2^{k-2}.
$$

We know that

$$
\nu(\Pi_{k-2}) = \nu(S) + \nu(\Pi_{k-3}) = \sum_{i=0}^{k-2} \delta_i,
$$

so

$$
\nu(T) - \nu(\Pi_{k-3}) \leq 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i.
$$

The join of $\{P\}$ and $\Pi_{k-2}$ is a $(k-1)$-space, intersecting $\Pi_{k-1}$ in a $(k-2)$-space, namely $T \cup \Pi_{k-3}$. Let $L_1$ and $L_2$ be the other two distinct $(k-2)$-spaces such that $\Pi_{k-3} \subset L_i \subset \Pi_{k-1}$ for $i = 1, 2$.

Now we have

$$
\sum_{i=0}^{k-1} \delta_i = \nu(\Pi_{k-1}) = \nu(L_1) + \nu(L_2) - \nu(\Pi_{k-3}) + \nu(T)
$$

$$
\leq 2 \left( \sum_{i=0}^{k-2} \delta_i - 1 \right) + 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i.
$$

This is simplified to

$$
\delta_{k-1} \leq 2^{k-2}\delta_1 - 2^{k-2} - 2,
$$

and the theorem is proved. $\Box$
2.3 Bounds on the total value

**Theorem 10 (Total value)** If $k \geq 2$, $1 \leq m \leq k-1$, and $(\delta_0, \delta_1, \ldots, \delta_k)$ satisfies $(N_{m-1} m)$, then

$$\nu(\text{PG}(k, q)) \leq \sum_{i=0}^{m-1} \delta_i + (\delta_m - 1) \sum_{i=0}^{k-m} q^i.$$  

**Proof.** Let $\alpha \in M_{m-1}$. In $\text{PG}(k, q)$ there are $\theta(k-m)$ $m$-spaces containing $\alpha$, and for every such $m$-space $\beta \supset \alpha$, we know by condition $(N_{m-1} m)$ that

$$\nu(\beta \setminus \alpha) \leq \delta_m - 1.$$  

Thus $\nu(\text{PG}(k, q) \setminus \alpha) \leq (\delta_m - 1) \theta(k-m)$. By the definition of $\alpha$, we know that

$$\nu(\alpha) = \sum_{i=0}^{m-1} \delta_i,$$

and the theorem follows. □

For an ENDS, several bounds may be derived from the above theorem. Corollary 11 is the best possible bound for $(k-1)$-optimal codes, while Corollary 12 is stronger for binary codes.

**Corollary 11** If $(\delta_0, \delta_1, \ldots, \delta_k)$ is a difference sequence satisfying $(N_{0.1})$ and $k \geq 2$, then

$$\nu(\text{PG}(k, q)) \leq \delta_0 + (\delta_1 - 1) \sum_{i=0}^{k-1} q^i \leq \sum_{i=0}^{k} q^i \delta_0 - (q + 2) \sum_{i=0}^{k-1} q^i.$$  

The bound holds with equality if and only if every line through $X \in M_0$ has value $(q + 1)\delta_0 - (q + 2)$.

**Corollary 12** If $(\delta_0, \delta_1, \ldots, \delta_k)$, $k \geq 2$, satisfies $(N_{k-2} k - 1)$, then $\delta_k \leq q\delta_{k-1} - (q + 1)$.

**Theorem 13** Let $2 \leq k \leq 4$. Then the given bounds on $\delta_1$ through $\delta_k$ are the best possible. In particular there exists a construction meeting the bounds with
equality if and only if the following constraint on \( \delta_0 \) is met

\[
\begin{align*}
\delta_0 &\geq 3 & \text{if } q = 2 & \land k = 2 \\
\delta_0 &\geq 5 & \text{if } q = 2 & \land k = 3 \\
\delta_0 &\geq 4 & \text{if } q = 2 & \land k = 4 \\
\delta_0 &\geq 2 & \text{if } q = 3 & \land k = 2 \\
\delta_0 &\geq 3 & \text{if } q = 3 & \land k = 3, 4 \\
\delta_0 &\geq 2 & \text{if } q \geq 4 & \land k = 2, 3 \\
\delta_0 &\geq 3 & \text{if } q \geq 4 & \land k = 4.
\end{align*}
\]

The theorem has been proved by giving explicit constructions. Chen and Kløve proved it for \( k = 3 \) and \( q \geq 3 \) in [2] and for \( k = 3 \) and \( q = 2 \) in [5]. It was proved for \( k = 4 \) in [12]. The example below shows it for \( k = 2 \). For \( k \leq 1 \), there are no non-chain conditions.

**Example 14** An optimal ENDS in \( \text{PG}(2, q) \) is easily obtained as follows. Let \( \ell \) be a line, and \( X \not\in \ell \) a point. Let \( \nu(X) = \delta_0 \). Consider each line \( \alpha \ni X \). If \( q \geq 3 \), we choose two points in \( \alpha \setminus (\{X\} \cup \ell) \) to have value \( \delta_0 - 2 \). All remaining points have value \( \delta_0 - 1 \). Note that \( \delta_0 \geq 2 \).

If \( q = 2 \), there is only one point in \( \alpha \setminus (\{X\} \cup \ell) \), so that point must have value \( \delta_0 - 3 \), thus \( \delta_0 \geq 3 \).

3 Structure of optimal codes

In this section we will find further necessary conditions for an extremal non-chain code to be \( m \)-optimal. For instance if \( H \in M_3 \) is a 3-space of maximum value, then there are a line \( \ell \subseteq H \) and a plane \( P \subseteq H \) such that

\[
\begin{align*}
\nu(p) &= \delta_0 - 3 & \forall p \in \ell \cap P \\
\nu(p) &= \delta_0 - 2 & \forall p \in \ell \cup P, \quad p \not\in \ell \cap P \\
\nu(p) &= \delta_0 - 1 & \text{otherwise}.
\end{align*}
\]

The general result is stated in Theorem 26.

**Lemma 15** If \( \delta_i = q\delta_{i-1} - 1 \) for \( i = 1, \ldots, k \), then

\[
\sum_{i=m}^k \delta_i = \theta(k-m)\delta_m - \sum_{i=0}^{k-m-1} \theta(i), \quad 0 \leq m \leq k.
\]
**Proof.** The equality follows immediately from the fact that if \(0 \leq i \leq m \leq k\), then
\[
\delta_m = q^i \delta_{m-i} - \theta(i - 1).
\]

\[ \square \]

**Lemma 16** If \(0 \leq a \leq q - 1\), then
\[
\theta(m) - a \sum_{i=0}^{m-1} \theta(i) \geq 1.
\]

**Proof.** We write
\[
\theta(m) - a \sum_{i=0}^{m-1} \theta(i) = \theta(m) - \frac{a}{q-1} \sum_{i=0}^{m-1} (q^{i+1} - 1)
\]
\[
= \theta(m) - \frac{a}{q-1} (\theta(m) - 1 - m) \geq 1.
\]

\[ \square \]

**Lemma 17** Let \(\nu\) be a value assignment with difference sequence \((\delta_0, \delta_1, \ldots, \delta_k)\) where \(\delta_i = q \delta_{i-1} - 1\) for \(i = 1, \ldots, k - 1\). If \(\Pi_m \subseteq M_m\), then \(\nu|_{\Pi_m}\) has difference sequence \((\delta_0, \delta_1, \ldots, \delta_m)\).

**Proof.** The proof is trivial for \(m = k\), so assume \(m < k\). Let
\[
\emptyset = \Theta_{-1} \subset \Theta_0 \subset \Theta_1 \subset \ldots \subset \Theta_m = \Pi_m
\]
be a chain of subspaces such that \(\Theta_i\) has the greatest value among the \(i\)-spaces containing \(\Theta_{i-1}\) in \(\Pi_m\). Define \(\delta_i' = \nu(\Theta_i) - \nu(\Theta_{i-1})\).

Let \((\delta_0', \delta_1', \ldots, \delta_m')\) be the difference sequence of \(\nu|_{\Pi_m}\). It is sufficient to prove that \(\delta_i' = \delta_i\) for all \(i\), because
\[
\sum_{i=0}^{j} \delta_i' \leq \sum_{i=0}^{j} \delta_i'' \leq \sum_{i=0}^{j} \delta_i, \quad 0 \leq j \leq m.
\]

\[ \text{(11)} \]

Suppose for contradiction that there is an \(i\) such that \(\delta_i \neq \delta_i'\). Let \(l\) be the smallest such \(i\). Note that \(\delta_i' < \delta_i\) by (11).

Since there are only \(\theta(m-l)\) distinct \(l\)-spaces containing \(\Theta_{l-1}\) in \(\Pi_m\), we get
\[
\nu(\Pi_m) \leq \theta(m-l) \delta_i' + \sum_{i=0}^{l-1} \delta_i' \leq \theta(m-l)(\delta_i - 1) + \sum_{i=0}^{l-1} \delta_i.
\]

13
Also note that by Lemma 15,
\[
\nu(\Pi_m) = \theta(m - l)\delta_l - \sum_{j=0}^{m-l-1} \theta(j) + \sum_{i=0}^{l-1} \delta_i.
\]

Combine the two lines to get
\[
\theta(m - l)\delta_l - \sum_{j=0}^{m-l-1} \theta(j) + \sum_{i=0}^{l-1} \delta_i \leq \theta(m - l)(\delta_l - 1) + \sum_{i=0}^{l-1} \delta_i,
\]
which is equivalent to
\[
\theta(m - l) - \sum_{j=0}^{m-l-1} \theta(j) \leq 0,
\]
contradicting Lemma 16. \(\square\)

**Corollary 18** Any code with difference sequence \((\delta_0, \delta_1, \ldots, \delta_k)\) such that \(\delta_i = q\delta_{i-1} - 1\) for \(i = 1, \ldots, k - 1\) satisfies the chain condition.

**Lemma 19** Let \(\nu\) be a value assignment with difference sequence \((\delta_0, \delta_1, \ldots, \delta_k)\) such that \(\delta_k = q\delta_{k-1}\). For every \((k - 1)\)-space \(\Pi_{k-1} \supset \Pi_{k-2} \in M_{k-2}\), we have \(\Pi_{k-1} \in M_{k-1}\).

**PROOF.** Consider \(\Pi_{k-2} \in M_{k-2}\). Let \(B_0, \ldots, B_q\) be the \((k - 1)\)-spaces such that \(\Pi_{k-2} \subset B_j, j = 0, \ldots, q\). We get
\[
\nu(\text{PG}(k, q)) = \sum_{j=0}^{q} \nu(B_j \setminus \Pi_{k-2}) + \nu(\Pi_{k-2}) = \sum_{j=0}^{k} \delta_j.
\]
Since \(\delta_k = q\delta_{k-1}\), we get that
\[
(q + 1)\delta_{k-1} = \sum_{j=0}^{q} \nu(B_j \setminus \Pi_{k-2}).
\]
Comparing this with the fact that \(\nu(B_j \setminus \Pi_{k-2}) \leq \delta_{k-1}\) for all \(j\), we get that \(B_j \in M_{k-1}\), as required. \(\square\)

We recall Corollary 7 and Remark 8 to get the following corollary.

**Corollary 20** If \((\delta_0, \delta_1, \ldots, \delta_k)\) is a 1-optimal ENDS, \(k \geq 2\), and \(\ell\) is line with value \(\nu(\ell) = \delta_0 + \delta_1\), then \(\nu(p) = \delta_0 - 1\) for all \(p \in \ell\).
Lemma 21 Let $\nu : \text{PG}(k,q) \to \mathbb{N}_0$ be a value assignment with difference sequence $(\delta_0, \delta_1, \ldots, \delta_k)$ such that $\delta_i = q\delta_{i-1} - 1$, $1 \leq i \leq k$. For every $\Pi_{m-1} \in M_{m-1}$, $0 \leq m \leq k$, we have that

(a) the number of distinct $m$-spaces of maximum value through $\Pi_{m-1}$ is at least

$$\theta(k - m) - \sum_{j=0}^{k-m-1} \theta(j).$$

(b) for $m = k - 1$ there is a unique $m$-space $\Pi_m \not\in M_m$ such that $\Pi_{m-1} \subset \Pi_m$, and

$$\nu(\Pi_m) = \sum_{j=0}^{m} \delta_j - 1.$$

**PROOF.** There are $\theta(k - m)$ $m$-spaces $B_i \supset \Pi_{m-1}$. We get that

$$\nu(\text{PG}(k,q)) = \sum_{j=1}^{\theta(k-m)} \nu(B_j \setminus \Pi_{m-1}) + \nu(\Pi_{m-1}) = \sum_{j=0}^{k} \delta_j.$$

and by Lemma 15,

$$\sum_{j=1}^{\theta(k-m)} \nu(B_j \setminus \Pi_{m-1}) = \sum_{j=m}^{k} \delta_j = \theta(k - m)\delta_m - \sum_{j=0}^{k-m-1} \theta(j).$$

Clearly

$$\nu(B_j \setminus \Pi_{m-1}) \leq \delta_m, \quad 1 \leq j \leq \theta(k - m). \quad (12)$$

Comparing the last two equations, we note that at least

$$\theta(k - m) - \sum_{j=0}^{k-m-1} \theta(j)$$

of the $B_i$ give equality in (12). If $m \leq k - 1$, then at least one of the $B_j$ gives inequality. The case where $m = k - 1$, is just a special case where $q$ of the $B_i$ gives equality and one gives inequality. The exact value of the one with inequality is easily computed. \qed

Lemma 22 Let $C$ be a code with difference sequence $(\delta_0, \delta_1, \ldots, \delta_k)$. If $\delta_i = q\delta_{i-1} - 1$ for $i = 1, \ldots, k$, then there exists at most one point which is not contained in any element of $M_{k-1}$.  

15
PROOF. Suppose there are two distinct points $P, Q \in PG(k, q)$ which are not contained in any element of $M_{k-1}$. Consider a chain

$$\Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{k-1} \subset PG(k, q),$$

such that $\Pi_i \subset M_i$ for each $i = 0, \ldots, k - 1$. Let $\ell$ span $P, Q$. Obviously there is a point $S \in \ell \cap \Pi_{k-1}$. By assumption $P, Q \notin \Pi_{k-1}$, so $S \neq P$ and $S \neq Q$.

We claim that we can assume that $S \notin \Pi_{k-2}$. By Lemma 21b, there are $q$ points in $\Pi_1$ which are elements of $M_0$, so if $S \in \Pi_0$, we can replace $\Pi_0$ by some other point which is in $\Pi_1$ and in $M_0$. For all $i$ such that $1 \leq i \leq k - 2$, there are $q$ $i$-spaces in $M_i$ containing $\Pi_{i-1}$ in $\Pi_{i+1}$. Thus if $S \in \Pi_i \setminus \Pi_{i-1}$, we can replace $\Pi_i$ with some other $i$-space, maintaining the chain. By induction we can assume that $S \notin \Pi_{k-2}$, as required.

There are $q + 1$ distinct $(k-1)$-spaces spanned by $\Pi_{k-2}$ and a point on $\ell$, and only one of these is not an element of $M_{k-1}$ by Lemma 21b. Since $(P)\Pi_{k-2}$ and $(Q)\Pi_{k-2}$ are two distinct $(k-1)$-spaces, either $P$ or $Q$ is contained in some element of $M_{k-1}$. The lemma follows by contradiction. □

Lemma 23 Let $\nu$ be a value assignment with difference sequence $(\delta_0, \delta_1, \ldots, \delta_k)$ such that $k \leq 2$ and $\delta_i = q\delta_{i-1} - 1$ for $1 \leq i \leq k$. Then there exists a collection $S$ containing exactly one $i$-space for each $i = 0, \ldots, k - 1$ such that

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in PG(k, q).$$

PROOF. For $k = 0$ the result is trivial.

For $k = 1$ there are $q + 1$ points. By Lemma 21b there is one point $P$ of value $\delta_0 - 1$ and $q$ points of value $\delta_0$. Hence $S = \{P\}$ forms the required collection.

Consider $k = 2$. There is a point $\varphi \in M_0$. Let $\ell_0, \ldots, \ell_q$ be the distinct lines such that $\varphi \subset \ell_i$ for all $i$. One of these lines, say $\ell_0$, has value $\delta_1 + \delta_0 - 1$, while the remaining $q$ lines have value $\delta_0 + \delta_1$ by Lemma 21b. This means that for $1 \leq i \leq q$, there is exactly one point $\alpha_i \in \ell_i$ such that $\nu(\alpha_i) = \delta_0 - 1$. There are at most two points in $\ell_0$ with value $\delta_0 - 1$ or less. The remaining points have value $\delta_0$. Obviously, every line in $PG(2, q)$ has value at most $\delta_0 + \delta_1$, and hence has at least one point of value $\delta_0 - 1$ or less. A set of $q + 2$ points cannot meet every line in a plane unless it contains a line [10, Lemma 13.4(iv)]. It follows that there must be a line $\Pi_1$ such that $\nu(p) \leq \delta_0 - 1$ for all $p \in \Pi_1$. Since $\nu(\ell_0) = \delta_1 + \delta_0 - 1$, there is either one point $\Pi_0 = \Pi_1 \cap \ell_0$ which has value $\delta_0 - 2$ or two distinct points $\Pi_0$ and $\Pi_1 \cap \ell_0$ of value $\delta_0 - 1$. In either case $\{\Pi_0, \Pi_1\}$ forms the required collection $S$. □
Definition 24 (Projections) We define the projection $\pi_p$ of $\text{PG}(k,q)$ through the point $p \in \text{PG}(k,q)$:

$$
\pi_p : \text{PG}(k,q) \to \text{PG}(k-1,q),
$$

by mapping distinct lines through $p$ in $\text{PG}(k,q)$ to distinct points in $\text{PG}(k-1,q)$ such that coplanar lines are taken to collinear points. We define the projected value assignment

$$
\nu_p : \text{PG}(k-1,q) \to \mathbb{N}_0,
$$

$$
\nu_p : X \mapsto \nu(\pi_p^{-1}(X)\setminus \{p\}).
$$

The code corresponding to $\nu_p$ is the subcode $\langle p \rangle^*$ of codimension 1 [7].

Lemma 25 Let $\nu : \text{PG}(k,q) \to \mathbb{N}_0$, $q \geq 3$, be the value assignment of a code $C$ with difference sequence $(\delta_0, \delta_1, \ldots, \delta_k)$ such that $\delta_i = q \delta_{i-1} - 1$ for $i = 1, \ldots, k$. Then there exists a collection $S$ containing exactly one $i$-space for each $i = 0, \ldots, k-1$ such that

$$
\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \text{PG}(k,q).
$$

PROOF. Lemma 23 proves it for $k < 3$. Now assume that the lemma holds for $k < n$, and consider

$$
\nu : \text{PG}(n,q) \to \mathbb{N}_0, \quad n \geq 3 \land q \geq 3.
$$

For $\Pi_k \in M_k$, $k < n$, let $S(\Pi_k)$ be the collection $S$ corresponding $\nu|_{\Pi_k}$. By Lemma 17 $\nu|_{\Pi_k}$ has difference sequence $(\delta_0, \delta_1, \ldots, \delta_k)$. Thus $S(\Pi_k)$ exists by the induction hypothesis, and it has the property given in the lemma. We define $\sigma_i(\Pi_k)$ to be the $i$-space in $S(\Pi_k)$.

Claim I If $\Theta_1 \in M_{n-2}$ and $\Theta_2 \in M_{n-1}$ such that $\Theta_1 \subset \Theta_2$, then

$$
\sigma_i(\Theta_1) = \Theta_1 \cap \sigma_{i+1}(\Theta_2), \quad 0 \leq i \leq n-3.
$$

We can use either $S(\Theta_1)$ or $S(\Theta_2)$ to express the value of a point $p \in \Theta_1$. Hence

$$
\#\{\Pi \in S(\Theta_1) \mid p \in \Pi\} = \#\{\Pi \in S(\Theta_2) \mid p \in \Pi\}. \quad (13)
$$

For all $i$, $\sigma'_i := \sigma_{i+1}(\Theta_2) \cap \Theta_1$ is either an $(i + 1)$-space if $\sigma_{i+1}(\Theta_2) \subseteq \Theta_1$, or else an $i$-space. Equation (13) can only be satisfied for all $p \in \Theta_1$ if $\dim \sigma'_i = i$ for all $i$. Hence we can let $\sigma'_i$ for $i \geq 0$ be the elements of $S(\Theta_1)$, and the claim follows.
Claim II If $1 \leq i \leq n - 2$, then there is an $(i + 1)$-space $\sigma_{i+1}$ such that $\sigma_i(A) \subset \sigma_{i+1}$ for all $A \in M_{n-i}$.

Consider $P \in M_{n-3}$, $\alpha_0 \in M_{n-2}$, $A_1, \ldots, A_q \in M_{n-1}$, and an $(n-1)$-space $A_0 \not\in M_{n-1}$ such that $P \subset A_0 \subset A_j$ for $0 \leq j \leq q$. Since $q \geq 3$, there are at least two distinct $(n-2)$-spaces $\alpha_1, \alpha_2 \in M_{n-2}$ such that $P \subset \alpha_j \subset A_1$ and $\alpha_0 \neq \alpha_j$ for $j = 1, 2$. There are also at least two distinct $(n-2)$-spaces $\beta_1, \beta_2 \in M_{n-2}$ such that $P \subset \beta_j \subset A_2$ and $\alpha_0 \neq \beta_j$ for $j = 1, 2$. Define $\sigma_{i+1}$ to meet $\alpha_1$ and $\beta_1$, and $\alpha_2$ and $\beta_2$. We have $A_1 \cap A_2 = \alpha_0 \in M_{n-2}$, so

$$\sigma_{i-1}(\alpha_0) = \sigma_i(A_1) \cap \alpha_0 = \sigma_i(A_2) \cap \alpha_0 = \sigma_i(A_1) \cap \sigma_i(A_2),$$

by Claim I. Since $\dim \sigma_{i-1}(\alpha_0) = i - 1$, we get $\dim \sigma_{i+1} = i + 1$. It remains to prove that $M_{n-1} = \mathcal{G}$ where

$$\mathcal{G} := \{A \in M_{n-1} \mid \sigma_i(A) \subset \sigma_{i+1}, 1 \leq i \leq n - 2\}.$$

Consider the spaces $\alpha_1 \beta_1$ and $\alpha_2 \beta_1$. At least one of them is a space in $M_{n-2}$, denote it $B_1$. Similarly, let $B_2$ be either $\alpha_1 \beta_2$ or $\alpha_2 \beta_2$ such that $B_2 \in M_{n-1}$. We have the following

$$B_1 \cap A_1 = \alpha_j \in M_{n-2}, \quad j = 1 \lor j = 2,$$
$$B_1 \cap A_2 = \beta_1 \in M_{n-2},$$
$$B_2 \cap A_1 = \alpha_j \in M_{n-2}, \quad j = 1 \lor j = 2,$$
$$B_2 \cap A_2 = \beta_2 \in M_{n-2}.$$

It follows that $\sigma_i(B_1) \cap \sigma_i(A_1) = \sigma_{i-1}(\alpha_j)$ for $j = 1$ or $j = 2$, and $\sigma_i(B_1) \cap \sigma_i(A_2) = \sigma_{i-1}(\beta_1)$. Hence $\sigma_i(B_1)$ meets $\sigma_{i+1}$ in two distinct $(i-1)$-spaces, and consequently $\sigma_i(B_1) \subset \sigma_{i+1}$. A similar argument holds for $B_2$, and hence $\sigma_i(B_2) \subset \sigma_{i+1}$.

At least one of the $(n-2)$-spaces $A_3 \cap B_1$ or $A_3 \cap B_2$ is an element $\alpha' \in M_{n-3}$, because $P = A_3 \cap B_1 \cap B_2 \in M_{n-3}$. It follows that $\sigma_i(A_3)$ meets $\sigma_{i+1}$ in at least two distinct $(i-1)$-spaces, $\sigma_{i-1}(\alpha')$ and $\sigma_{i-1}(\alpha_0)$. We conclude that $\sigma_i(A_3) \subset \sigma_{i+1}$. So far we have shown that

$$A_1, A_2, A_3, B_1, B_2 \in \mathcal{G}.$$

We note that if there are two distinct elements $E_1, E_2 \in \mathcal{G}$, and $A \in M_{n-1}$ such that $\gamma_j \in E_j \cap A \in M_{n-2}$ for $j = 1, 2$ and $\gamma_1 \neq \gamma_2$, then $\sigma_i(A)$ meets $\sigma_{i+1}$ in two distinct $(i-1)$-spaces $\sigma_{i-1}(\gamma_j)$. Hence $A \in \mathcal{G}$.
If there are three distinct elements \( E_1, E_2, E_3 \in \mathcal{S} \) and \( A \in M_{n-3} \) such that the intersections \( E_j \cap A \) are three distinct \((n - 2)\)-spaces and

\[
A \cap \bigcap_{j=1}^{3} E_j \in M_{n-3},
\]

then at least two of the \( E_j \) meets \( A \) in distinct elements of \( M_{n-2} \), and \( A \in \mathcal{S} \).

Consider an element \( A \in M_{n-1} \) such that

\[
P \subset A \not\in \{A_1, A_2, A_3, B_1, B_2\}.
\]

If \( a_0 \not\subset A \), then \( A \) meets \( A_1, A_2, \) and \( A_3 \) in three distinct \((n - 2)\)-spaces containing \( P \), and thus \( A \in \mathcal{S} \). If \( a_0 \subset A \), then \( A \) meets \( A_1, B_1, \) and \( B_2 \) in three distinct \((n - 2)\)-spaces containing \( P \) and \( A \in \mathcal{S} \). Thus we have proved that if \( P \subset A \in M_{n-1} \), then \( A \in \mathcal{S} \).

If \( A \in M_{n-1} \) such that \( \bar{P} \cap A \in M_{n-4} \), then there is \( \xi \in M_{n-2} \) such that \( P \subset \xi \) and \( S := \xi \cap A \in M_{n-3} \). This is obvious from the fact that there are at least \( q^2 - 1 \) \((n - 2)\)-spaces of maximum value through \( P \) by Lemma 21, and at most \( q + 2 \) \((n - 3)\)-spaces through \( \bar{P} \) in \( A \) that are not elements of \( M_{n-3} \). Hence there are at least \( q^2 - q - 3 \geq 3 \) choices for \( \xi \). There are at least three subspaces \( E_j \in M_{n-1}, j = 1, 2, 3 \), through \( \xi \), and

\[
A \cap \bigcap_{j=1}^{3} E_j = S \in M_{n-3}.
\]

Hence \( A \in \mathcal{S} \).

Suppose for induction that if \( P \not\subset A \in M_{n-1} \) and there is \( R \subset \bar{P} \cap A \) such that \( R \in M_{j+1} \), then \( A \in \mathcal{S} \). This was proved for \( j = n - 5 \) in the last paragraph. It even holds when \( n = 3 \), because if \( j = -2 \), then \( R = \emptyset \in M_{-1} \).

Consider \( A \in M_{n-1} \) such that there is \( \bar{R} \in M_j \) such that \( \bar{R} \subset \bar{P} \), but there is no \( \bar{R}' \in M_{j+1} \) such that \( \bar{R}' \subset \bar{P} \). Let \( R \in M_{j+1} \) be such that \( R \subset \bar{R} \subset \bar{P} \). We shall prove that there is \( \xi \in M_{n-2} \) such that \( R \subset \xi \) and \( \xi \cap A \in M_{n-3} \). This is sufficient because then there are \( q \geq 3 \) elements of \( \mathcal{S} \) containing \( \xi \) by the induction hypothesis, and at least two of them meet \( A \) in elements of \( M_{n-2} \).

We prove the existence of \( \xi \) by induction on \( m \). Assume that

\[
\exists R_m \in M_m, \ s.t. \ R_m \cap A \in M_{m-1}, \ j + 1 \leq m \leq n - 3. \quad (14)
\]

Let \( R_{j+1} = R \). By Lemma 21, there are at least

\[
\theta(n - (m + 1)) - \sum_{l=0}^{n-(m+1)-1} \theta(l)
\]

19
\((m + 1)\)-spaces of maximum value through \(R_m\). Of these at most
\[
\sum_{l=0}^{n-1-m-1} \theta(l)
\]
meet \(A\) in an \(m\)-space which does not have maximum value. Hence at least
\[
\theta(n - m - 1) - 2 \sum_{l=0}^{n-m-2} \theta(l) \geq 1
\]
\((m + 1)\)-spaces satisfy (14) by Lemma 16. By induction \(\xi R_{n-2}\) exists, and hence \(\mathcal{S} = M_{n-1}\). This proves Claim II.

**Claim III** For all \(A \in M_{n-1}\), \(1 \leq i \leq n - 2\), \(\sigma_i(A) = \sigma_{i+1} \cap A\).

By the previous claim it is sufficient to prove that \(\sigma_{i+1} \not\subseteq A\). Assume for contradiction that the claim fails for some \(i\), and let \(m\) be the largest such \(i\). Let \(A \in M_{n-1}\) be such that \(\sigma_{m+1} \subseteq A\). Let \(B \in M_{n-1}\) such that \(\sigma_{m}(A) \neq \sigma_{m}(B)\). By Claim II we get that \(\sigma_{m}(B) \subset \sigma_{m+1} \subseteq A\). Note that
\[
\#\sigma_{m}(B) = \theta(m)
\]
\[
\#(\sigma_{m}(A) \cap \sigma_{m}(B)) \leq \theta(m - 1)
\]
\[
\# \bigcup_{j=0}^{m-1} \sigma_{j}(A) \leq \sum_{j=0}^{m-1} \theta(j).
\]
Hence
\[
\# \left( \sigma_{m}(B) \setminus \bigcup_{i=0}^{m} \sigma_{i}(A) \right) \geq q^{m} - \sum_{j=0}^{m-1} \theta(j) \geq 1,
\]
since \(q \geq 3\). It follows that there exists
\[
p \in \sigma_{m}(B) \setminus \bigcup_{i=0}^{m} \sigma_{i}(A).
\]
Since the claim is assumed to hold for \(i > m\), we have that
\[
\nu(p) = \delta_{0} - \# \{ i \mid p \in \sigma_{i}(B) \land 0 \leq i \leq n - 2 \}
\]
\[
\leq \delta_{0} - 1 - \# \{ i \mid p \in \sigma_{i+1} \land m + 1 \leq i \leq n - 2 \}
\]
\[
\nu(p) = \delta_{0} - \# \{ i \mid p \in \sigma_{i}(A) \land 0 \leq i \leq n - 2 \}
\]
\[
= \delta_{0} - \# \{ i \mid p \in \sigma_{i+1} \land m + 1 \leq i \leq n - 2 \},
\]
and these two equations contradict each other, proving Claim III.

We write
\[
U := \{ \sigma_{0}(A) \mid A \in M_{n-1} \}.
\]
Lemma 22 says that at most one point is not contained in any element of $M_{n-1}$. This means that we can form the set

$$S' = U \cup \{\sigma_i \mid i = 2, \ldots, n-1\},$$

giving the value of all points but at most one by the formula

$$\nu(p) = \delta_0 - \#\{\Pi \in S' \mid p \in \Pi\}.$$

**Claim IV** There is a line $\sigma_1$ such that $\sigma_0(A) \subset \sigma_1$ for all $A \in M_{n-1}$.

Take a point $\{F\} \in M_0$ such that

$$F \in \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{n-3} = P$$

is a chain of subspaces of maximum value. The projected value assignment $\nu_F$ defines an $(n-1)$-dimensional subcode code with weight $d_{n-1}$. The difference sequence of $\nu_F$ is $(\delta_1, \ldots, \delta_n)$, because $\pi_F(\Pi_i) \in M_{i-1}(\nu_F)$ for $0 \leq i \leq n$.

By the induction hypothesis, there is a collection $S(PG(n-1,q))$ of $i$-spaces $\sigma_i(PG(n-1,q))$ for $i = 0, \ldots, n-2$ such that

$$\nu_F(p) = \delta_1 - \#\{\Pi \in S(PG(n-1,q)) \mid p \in \Pi\}.$$

Clearly $F \not\in \Pi$ for any $\Pi \in S'$. Hence $\pi_F(\sigma_i)$ is an $i$-space. We get the following formula for the values of every point but at most one in $PG(n-1,q)$:

$$\nu_F(p) = q\delta_0 - \#\{\Pi \in S' \mid p \in \pi_F(\Pi)\}
= \delta_1 - \#\{\Pi \in S' \setminus \{\sigma_{n-1}\} \mid p \in \pi_F(\Pi)\}.$$

It follows that

$$\pi_F(\sigma_i) = \sigma_i(PG(n-1,q)), \quad 2 \leq i \leq n-2$$

$$\pi_F(U) \subseteq \sigma_1(PG(n-1,q)) \cup \sigma_0(PG(n-1,q)).$$

We have $U \cap \alpha_0 = \emptyset$ by Claim I. It follows that $\sigma_0(A_i)$ for $i = 1, \ldots, q$ are $q$ distinct elements of $U$. Let $U' = U \setminus A_0$ be the set of these $q$ points.

Now consider $V = \sigma_1(PG(n-1,q)) \cup \sigma_0(PG(n-1,q))$, the inverse image of which must consist of points in $U$ and points not contained in any element of $M_{n-1}$. In fact $\pi_F(U') \subset \sigma_1(PG(n-1,q))$. Hence $U'$ are coplanar points.

There are more chains

$$F \neq F' \in \Pi'_0 \subset \Pi'_1 \subset \ldots \subset \Pi'_{n-3} \subset \alpha_0$$

of subspaces of maximum value. By projecting through such a point $F'$, we can show that $U'$ is also contained in a plane which is not equal to the first. Hence $U'$ is contained in a line, which we denote $\sigma_1$, and $\pi_F(\sigma_1) = \sigma_1(PG(n-1,q))$.
We shall prove that \( U \cap A_0 \subset \sigma_1 \), and consequently that \( U \subseteq \sigma_1 \). This is trivial if \( U \cap A_0 = \emptyset \). Otherwise consider an arbitrary point \( R \in U \cap A_0 \). By the definition of \( U \), there is \( G \in M_{n-1} \) such that \( R \in G \). By Lemma 17 there is a subspace \( \rho \subset G \) such that \( \rho \in M_{n-2} \). By the argument used to prove Lemma 22, we can choose \( \rho \) such that \( R \not\in \rho \). Projecting through a couple of distinct points contained in \( M_0 \) and in \( \rho \), as we did in the previous paragraph, will show that \( R \in \sigma_1 \), as required. This proves Claim IV.

**Claim V** There is a point \( \sigma_0 \) which is not contained in any element of \( M_{n-1} \), and \( S := \{ \sigma_i \mid i = 0, \ldots, n-1 \} \) forms the required collection such that

\[
\nu(p) = \delta_0 - \#\{ \Pi \in S \mid p \in \Pi \}, \quad \forall p \in \Pi_n.
\]  

First assume that \( \sigma_0 \) does exist. We have proved that (15) holds for all points except possibly for \( \sigma_0 \). If it does fail for \( \sigma_0 \), it must give us a wrong value for \( \nu(\text{PG}(n, q)) \), but

\[
\nu(\text{PG}(n, q)) = \theta(n)\delta_0 - \sum_{\Pi \in S} \#\Pi = \theta(n)\delta_0 - \sum_{i=0}^{n-1} \theta(i) = \sum_{i=0}^{n} \delta_i,
\]

by Lemma 15, and that is correct. If \( \sigma_0 \) did not exist, we would have no point in \( S \), and the total value would not be correct. This completes the proof of Claim V and the lemma. \( \square \)

**Theorem 26** Let \( C \) be a chained, non-binary code with difference sequence \((\delta_0, \delta_1, \ldots, \delta_k)\). If

\[
\delta_i = q\delta_{i-1} - 1, \quad i = 1, \ldots, k-1,
\]

\[
\delta_k = q\delta_{k-1},
\]

then there exists a collection \( S \) of exactly one \( i \)-space in \( \text{PG}(k, q) \) for each \( i = 1, \ldots, k-1 \), such that

\[
\nu(p) = \delta_0 - \#\{ \Pi \in S \mid p \in \Pi \}, \quad \forall p \in \text{PG}(k, q).
\]

**Proof.** Lemma 25 says that for each \( \Pi_{k-1} \in M_{k-1} \), there is a set \( S(\Pi_{k-1}) \) such that

\[
\nu(p) = \delta_0 - \#\{ \Pi \in S(\Pi_{k-1}) \mid p \in \Pi \}, \quad \forall p \in \Pi_{k-1}.
\]

Let \( \sigma_i \) denote the \( i \)-space in \( S \). If \( k \geq 3 \) we use the same argument as in the proof of Lemma 25, to show that

\[
\sigma_i = \bigcup_{\Pi \in M_{k-1}} \sigma_{i-1}(\Pi), \quad i = 1, 2, \ldots, k-1.
\]

22
Because every point is contained in some $\Pi_{k-1} \in M_{k-1}$, there is no point in $S$.

The cases for $k \leq 2$ are just as simple as the proof of Lemma 23. \hfill \square

This theorem will of course apply to every subspace $\Pi_m \in M_m(C)$ for an $m$-optimal, extremal non-chain code $C$, and this fact has been most useful to limit the search for $m$-optimal constructions.

**Corollary 27** If $(\delta_0, \delta_1, \ldots, \delta_k)$ is a 3-optimal ENDS where $k \geq 4$ and $q \geq 3$, then $\delta_0 \geq 3$.

**Proof.** Let $\Pi_3 \in M_3$, and apply the theorem on $\nu|_{\Pi_3}$. There is $p \in \Pi_3$, such that $\nu(p) = (\delta_0 - 1) - 2$. \hfill \square

## 4 Acknowledgement

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## References


