# Upper bounds on Weight Hierarchies of Extremal Non-Chain Codes

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**Abstract**

The weight hierarchy of a linear $[n, k; q]$ code $\mathcal{C}$ over $\mathsf{GF}(q)$ is the sequence $(d_1, d_2, \dots, d_k)$ where $d_r$ is the smallest support weight of an $r$-dimensional subcode of $\mathcal{C}$. Linear codes may be classified according to a set of chain and non-chain conditions, the extreme cases being codes satisfying the chain condition (due to Wei and Yang) and extremal, non-chain codes (due to Chen and Kløve). This paper gives upper bounds on the weight hierarchies of the latter class of codes.

*Key words:* Weight hierarchy, chain condition, linear codes, projective multiset

## 1 Introduction

The concept of generalised Hamming weights was introduced as early as 1977 by Helleseth et al. [8] in their study of weight distributions of irreducible cyclic codes. The term 'generalised Hamming weight' was introduced by Wei in 1991 [14]. He used the parameters to analyse an application of codes on the Wire-Tap Channel of type II, which had been introduced in 1984 by Ozarow and Wyner [11]. During the nineties, several researchers have studied the generalised Hamming weights of linear codes.

The chain condition was introduced by Wei and Yang [15]. Chen and Kløve [2] introduced the opposite extreme, extremal non-chain codes. Known codes with high generalised Hamming weights tend to satisfy the chain condition. Cohen et al. [6] argue that some non-chain codes may have other advantages. Our interest is purely mathematical however.

Chen and Kløve found tight upper bounds for non-binary, four-dimensional, extremal non-chain codes [2]. Later they have also found all possible weight hierarchies of four-dimensional binary codes [5]. In this paper we generalise their upper bounds to arbitrary dimension, and these bounds are the best possible in dimension 5 and lower.

### 1.1 Notation and definitions

Throughout this paper $\mathcal{C}$ will denote an $[n, k+1; q]$ code, i.e. a linear code of length $n$ and dimension $k+1$ over the Galois field $\mathsf{GF}(q)$ with $q$ elements. Codes of dimension $k+1$ will be studied in a projective space $\mathsf{PG}(k, q)$ of dimension $k$ and order $q$.

Given a code $\mathcal{C}$ we define the support $\chi(\mathcal{C})$ to be the set of positions where not all codewords of $\mathcal{C}$ are zero, i.e.

$$\chi(\mathcal{C}) := \{i \mid \exists(x_1, x_2, \ldots, x_n) \in \mathcal{C}, \ \text{s.t. } x_i \neq 0\}.$$

The *support weight* of $\mathcal{C}$ is the size of $\chi(\mathcal{C})$, and we denote it $w_S(\mathcal{C})$, i.e.

$$w_S(\mathcal{C}) := \#\chi(\mathcal{C}).$$

For $0 \leq r \leq k+1$, *the* $r$*th generalised Hamming weight* $d_r$ of $\mathcal{C}$ is the least support weight of an $r$-dimensional subcode of $\mathcal{C}$. The sequence $(d_1, d_2, \ldots, d_{k+1})$ is called the weight hierarchy of $\mathcal{C}$. The minimum weight of the code is $d = d_1$.

We note that by adding a zero-position to $\mathcal{C}$, we get an $[n+1, k+1; q]$ code with the same weight hierarchy as $\mathcal{C}$. Without loss of generality, we can restrict our study to codes without zero-positions. In other words, we assume that $d_{k+1} = n$.

Two linear codes are *equivalent* if one can be obtained from the other by permuting coordinate positions or by multiplying some coordinate by a non-zero scalar. We note that equivalent codes have the same weight hierarchy.

### 1.2 Codes in projective geometry

We let $G$ denote a generator matrix of $\mathcal{C}$. The *value* (or multiplicity) $\nu(\mathbf{x})$ of $\mathbf{x} \in \mathsf{GF}(q)^{k+1}$ is the number of occurrences of $\mathbf{x}$ as a column in $G$. Replacing some column $\mathbf{x}$ with $a\mathbf{x}$ for some non-zero scalar $a$ leads to an equivalent code. Thus we can consider the columns of $G$ to be projective points, and an equivalence class of codes is uniquely determined by giving the map

$$\nu : \mathsf{PG}(k, q) \longrightarrow \mathbb{N}_0 := \{0, 1, \ldots\}.$$

This concept has been studied by several authors using different terminology. Dodunekov and Simonis [7] give an historic overview, and they prefer to call $\nu$ a projective multiset. In this paper we prefer to call it a *value assignment*, as did Chen and Kløve [2]. Tsfasman and Vladuţ [13] studied an equivalent concept called a projective system.

An arbitrary map $\nu : \mathsf{PG}(k,q) \longrightarrow \mathbb{N}_0$ is called a value assignment even if it is not defined from a code. We call it *non-degenerate* if there are $k+1$ projectively independent points $p_0, p_1, \ldots, p_k$ such that $\nu(p_i) \geq 1$ for all $i$. By taking $\nu(\mathbf{x})$ not necessarily distinct representatives for each projective point and taking an ordering on all these representatives, we get a matrix $G$. This matrix $G$ is a generator matrix of a code if and only if its rank is $k+1$, that is if $\nu$ is non-degenerate.

We define the value of a set of points as follows

$$\nu(U) := \sum_{\mathbf{x} \in U} \nu(\mathbf{x}), \quad \forall\, U \subseteq \mathsf{PG}(k,q).$$

Let $\mathsf{PG}^{(m)}(k,q)$ be the set of $m$-*spaces* or $m$-dimensional subspaces of $\mathsf{PG}(k,q)$. Note that $\mathsf{PG}^{(0)}(k,q)$ is the collection of subsets of cardinality 1; both $P \in \mathsf{PG}(k,q)$ and $\{P\} \in \mathsf{PG}^{(0)}(k,q)$ will be called a point. The 1-, 2- and $(k-1)$-spaces are called lines, planes, and hyperplanes, respectively. The only $(-1)$-space is the empty set.

The join of $\Pi_r$ and $\Pi_s$, denoted $\Pi_r\Pi_s$, is the intersection of all subspaces containing the union $\Pi_r \cup \Pi_s$. If $p_0, p_1, \ldots, p_m \in \mathsf{PG}(k,q)$ are projectively independent points, we write $\langle p_0, p_1, \ldots, p_m \rangle$ for their join. We define the following shorthand notation,

$$\theta(n) := \frac{q^{n+1}-1}{q-1} = \sum_{i=0}^{n} q^i,$$

and recall that $\theta(k)$ is the cardinality of $\mathsf{PG}(k,q)$.

### *1.3 Subcodes and the value assignments*

From now on we let $\nu : \mathsf{PG}(k,q) \to \mathbb{N}_0$ be the value assignment corresponding to $\mathcal{C}$. There is a one-to-one correspondence between subcodes of $\mathcal{C}$ of dimension $r$ and subspaces of $\mathsf{PG}(k,q)$ of dimension $k-r$. We write $\mathcal{D}^*$ for the projective subspace corresponding to a subcode $\mathcal{D} \subseteq \mathcal{C}$, and $\Pi^*$ for the subcode corresponding to $\Pi \subseteq \mathsf{PG}(k,q)$. If $\mathcal{D}_1 \subseteq \mathcal{D}_2$, then $\mathcal{D}_1^* \supseteq \mathcal{D}_2^*$. It is known [9,13] that $d_{k+1} - w_S(\mathcal{D}) = \nu(\mathcal{D}^*)$.

We define the weight hierarchy $(d_1, \ldots, d_{k+1})$ of a value assignment $\nu$ by letting

$n - d_r$ be the greatest value of a subspace of codimension $r$ in $\mathsf{PG}(k,q)$. Obviously the correspondence between value assignments and codes preserves the weight hierarchy. Note that a value assignment is non-degenerate if and only if $d_1 > 0$. All value assignments encountered in this paper are non-degenerate.

The difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ of a code or of a value assignment is defined by

$$\delta_j := d_{k+1-j} - d_{k-j}, \quad j = 0, 1, \ldots, k.$$

We note that the difference sequence is easily computed from the weight hierarchy and vice versa. We say that the difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ has dimension $k+1$. The elements of the difference sequence of a code or non-degenerate value assignment are positive, due to the strict monotonicity of the generalised Hamming weights.

The existence of a linear code with weight hierarchy $(d_1, d_2, \ldots, d_{k+1})$ is equivalent to the existence of a non-degenerate value assignment $\nu$ such that,

$$\max\{\nu(\Pi_m) \mid \Pi_m \in \mathsf{PG}^{(m)}(k,q)\} = \sum_{i=0}^{m} \delta_i, \quad -1 \leq m \leq k.$$

The set of $m$-spaces of maximum value is denoted by $M_m$,

$$M_m(\nu) := \left\{ \Pi_m \;\middle|\; \Pi_m \in \mathsf{PG}^{(m)}(k,q) \;\wedge\; \nu(\Pi_m) = \sum_{i=0}^{m} \delta_i \right\}, \quad -1 \leq m \leq k.$$

When no ambiguity is expected, we write $M_m = M_m(\nu)$.

Given an $m$-space $\Pi_m \in \mathsf{PG}^{(m)}(k,q)$, we can restrict the value assignment $\nu$ to this subspace and study

$$\nu' = \nu|_{\Pi_m} : \; \Pi_m \to \mathbb{N}_0.$$

If $\Pi_m \in M_m(\nu)$, the monotonicity of the weight hierarchy ensures that any proper subspace of $\Pi_m$ has lower value. In this case $\nu'$ is non-degenerate, and thus defines a code $\mathcal{D}$, which is actually the code obtained by puncturing $\mathcal{C}$ on each coordinate in $\chi(\Pi_m^*)$. We write $M_i(\Pi_m)$ $M_i(\nu|_{\Pi_m})$ for $-1 \leq i \leq m$.

### *1.4 The Chain Condition*

The chain condition was introduced by Wei and Yang [15], and it says that

$$\forall\, i \text{ s.t. } 0 \leq i \leq k-1 \quad \exists\, \Pi_i \in M_i \quad \text{s.t. } \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{k-1}.$$

We will refer to codes satisfying this condition as *chained codes*.

We define a number of subconditions, which are implications of the chain condition. For all $i$ and $j$ such that $0 \leq i < j \leq k-1$, we have the condition,

$$(\mathrm{C}i.j): \quad \exists \Pi_i \in M_i \; \exists \Pi_j \in M_j \;\text{ s.t. } \Pi_i \subset \Pi_j.$$

The negations of these conditions, $(\mathrm{N}i.j) := \neg(\mathrm{C}i.j)$, will be called *non-chain conditions*.

Analogous to the definition by Chen and Kløve [2], we define extremal non-chain codes of arbitrary dimension to be codes that satisfy all of the non-chain conditions $(\mathrm{N}i.j)$. The difference sequence of an extremal non-chain code will be called an ENDS (*extremal non-chain difference sequence* ).

## 2 Upper bounds

### *2.1 The general upper bound*

**Theorem 1** *If* $(\delta_0, \delta_1, \ldots \delta_k)$ *is an ENDS and* $1 \leq m \leq k-1$*, then*

$$\delta_m \leq q^m \delta_0 - \sum_{i=0}^{m} q^i.$$

*If this bound holds with equality for* $m = \bar{m} > 1$*, then it also holds with equality for* $m = \bar{m} - 1$*.*

The proof of this theorem is quite tedious, and we have to start with some auxiliary results.

**Definition 2** *We say that an ENDS is* $m$-optimal*,* $1 \leq m \leq k-1$*, if it satisfies the bound on* $\delta_m$ *from Theorem 1 with equality. An extremal non-chain code* $C$ *is* $m$-optimal *if its difference sequence is an* $m$*-optimal ENDS.*

**Lemma 3** *Given an arbitrary code with difference sequence* $(\delta_0, \delta_1, \ldots \delta_k)$*, we have* $\delta_k \leq q\delta_{k-1}$*.*

**PROOF.** Take some $\Pi_{k-2} \in M_{k-2}$. There are $q+1$ $(k-1)$-spaces through $\Pi_{k-2}$, and for every such subspace $\Pi_{k-1}$ we have

$$\nu(\Pi_{k-1} \backslash \Pi_{k-2}) \leq \delta_{k-1}.$$

The geometry is partitioned into $q+1$ disjoint subsets of the form $\Pi_{k-1} \backslash \Pi_{k-2}$, beside $\Pi_{k-2}$. Hence

$$\sum_{i=0}^{k} \delta_i \leq (q+1)\delta_{k-1} + \sum_{i=0}^{k-2} \delta_i.$$

The lemma follows immediately. $\square$

**Lemma 4** *Let $(\delta_0, \delta_1, \ldots \delta_k)$ be the difference sequence of some non-degenerate value assignment $\nu$, and $(\delta'_0, \ldots, \delta'_{k-1})$ the difference sequence of $\nu|_{\Pi_{k-1}}$ for some $\Pi_{k-1} \in M_{k-1}$. Then $\delta_{k-1} \leq \delta'_{k-1}$.*

**PROOF.** We have $\Pi_{k-1} \in M_{k-1}(\Pi_{k-1}) \subseteq M_{k-1}(\nu)$. Let $\Pi_{k-2} \in M_{k-2}(\nu)$ and $\Pi'_{k-2} \in M_{k-2}(\Pi_{k-1})$. Clearly $\nu(\Pi'_{k-2}) \leq \nu(\Pi_{k-2})$. Hence

$$\delta_{k-1} = \nu(\Pi_{k-1}) - \nu(\Pi_{k-2}) \leq \nu(\Pi_{k-1}) - \nu(\Pi'_{k-2}) = \delta'_{k-1},$$

as required. $\square$

**Lemma 5** *Let $\nu$ be the value assignment of an extremal non-chain code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. If $\Pi_m \in M_m$ where $0 \leq m \leq k$ and $\nu|_{\Pi_m}$ has difference sequence $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m)$, then $\delta_m \leq \varepsilon_m - 1$.*

**PROOF.** This goes almost like the proof of Lemma 4, except that since the code is extremal non-chain, we get a stronger bound. We have $\Pi_m \in M_m(\Pi_m) \subseteq M_m(\nu)$. Let $\Pi_{m-1} \in M_{m-1}(\nu)$ and $\Pi'_{m-1} \in M_{m-1}(\Pi_m)$. Since the code is extremal non-chain, we have $\nu(\Pi'_m) < \nu(\Pi_m)$. Hence

$$\delta_m = \nu(\Pi_m) - \nu(\Pi_{m-1}) \leq \nu(\Pi_m) - (\nu(\Pi'_{m-1}) + 1) = \varepsilon_m - 1,$$

as required. $\square$

**Lemma 6** *If $k \geq 2$ and $(\delta_0, \delta_1, \ldots \delta_k)$ satisfies* (N0.1)*, then $\delta_1 \leq q\delta_0 - (q+1)$ and $\delta_0 \geq 2$.*

**PROOF.** A line consists of $q+1$ points, and by (N0.1), $\delta_1 + \delta_0 \leq (q+1)(\delta_0 - 1)$. Hence $\delta_1 \leq q\delta_0 - (q+1)$. Also if $\delta_0 \leq 1$, then $\delta_1 \leq -1$, which is absurd. $\square$

**Proof of Theorem 1.** The proof goes by induction on $m$, so we assume that the theorem holds for every $m < \bar{m}$. Lemma 6 proves it for $m = 1$. Now we consider a code $\mathcal{C}$ such that

$$\delta_{\bar{m}} \geq q^{\bar{m}}\delta_0 - \theta(\bar{m}) \tag{1}$$
$$\delta_m \leq q^m\delta_0 - \theta(m), \quad \forall m \leq \bar{m} - 1. \tag{2}$$

Our aim is to prove that then we must have equality both in (1) and in (2).

Take an arbitrary $\Theta_{\bar{m}} \in M_{\bar{m}}(\mathcal{C})$, and let

$$\Theta_0 \subset \Theta_1 \subset \ldots \subset \Theta_{\bar{m}-1} \subset \Theta_{\bar{m}}$$

be a chain such that $\Theta_i \in M_i(\Theta_{i+1})$ for $0 \leq i \leq \bar{m}-1$. Let $(\varepsilon_0^{(i)}, \ldots, \varepsilon_i^{(i)})$ be the difference sequence of $\nu|_{\Theta_i}$.

By Lemma 5 and (1), we get

$$\varepsilon_{\bar{m}}^{(\bar{m})} \geq \delta_{\bar{m}} + 1 \geq q^{\bar{m}}\delta_0 - \theta(\bar{m}) + 1. \tag{3}$$

Lemma 4 and 3 give

$$\varepsilon_{\bar{m}-1}^{(\bar{m}-1)} \geq \varepsilon_{\bar{m}-1}^{(\bar{m})} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q} \right\rceil. \tag{4}$$

Repeating this argument $\bar{m}$ times and substituting from (3), we obtain

$$\varepsilon_0^{(0)} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q^{\bar{m}}} \right\rceil \geq \left\lceil \frac{q^{\bar{m}}\delta_0 - \theta(\bar{m}) + 1}{q^{\bar{m}}} \right\rceil = \delta_0 - 1.$$

Clearly $\varepsilon_0^{(0)}$ is the value of $\Theta_0$, which is a point in $\Theta_{\bar{m}} \in M_{\bar{m}}(\mathcal{C})$. Since $\mathcal{C}$ is extremal non-chain, we have $\varepsilon_0^{(0)} \leq \delta_0 - 1$. We conclude that

$$\varepsilon_0^{(l)} = \nu(\Theta_0) = \delta_0 - 1, \quad \forall l,\ 0 \leq l \leq \bar{m}. \tag{5}$$

We assume for induction on $j$ that for all $j < i$ where $0 < i < \bar{m}$, we have

$$\varepsilon_j^{(l)} = \varepsilon_j^{(j)} = q^j\delta_0 - \theta(j), \quad \forall l, \text{ s.t. } j \leq l \leq \bar{m}. \tag{6}$$

First we prove that it also holds for $l = j = i$. Repeating the argument of (4) $\bar{m} - i$ times, we get

$$\varepsilon_i^{(i)} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q^{\bar{m}-i}} \right\rceil \geq \left\lceil \frac{q^{\bar{m}}\delta_0 - \theta(\bar{m}) + 1}{q^{\bar{m}-i}} \right\rceil = q^i\delta_0 - \theta(i). \tag{7}$$

Now $\varepsilon_i^{(i)} = \nu(\Theta_i) - \nu(\Theta_{i-1})$. Since $\mathcal{C}$ is extremal non-chain, we get by (2) that

$$\nu(\Theta_i) \leq \sum_{j=0}^{i} \delta_j - 1 \leq \sum_{j=0}^{i} [q^j\delta_0 - \theta(j)],$$

and according to the induction hypothesis (6), we have

$$\nu(\Theta_{i-1}) = \sum_{j=0}^{i-1} \varepsilon_j^{(i-1)} = \sum_{j=0}^{i-1} \varepsilon_j^{(j)} = \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)]. \tag{8}$$

Combining these expressions, we get an upper bound on $\varepsilon_i^{(i)}$:

$$\begin{aligned} \varepsilon_i^{(i)} &= \nu(\Theta_i) - \nu(\Theta_{i-1}) \\ &\leq \sum_{j=0}^{i} [q^j \delta_0 - \theta(j)] - \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)] = q^i \delta_0 - \theta(i). \end{aligned} \tag{9}$$

Combining the upper and lower bounds (7) and (9), we conclude by induction that

$$\varepsilon_i^{(i)} = q^i \delta_0 - \theta(i), \quad i = 0, \ldots, \bar{m} - 1. \tag{10}$$

From (8) and (2) we can see that

$$\sum_{j=0}^{i-1} \delta_j - 1 \geq \nu(\Theta_{i-1}) = \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)] \geq \sum_{j=0}^{i-1} \delta_j - 1,$$

Hence $\delta_{i-1} = q^{i-1} \delta_0 - \theta(i-1)$. Also

$$\varepsilon_i^{(i)} + \nu(\Theta_{i-1}) = q^i \delta_0 - \theta(i) + \nu(\Theta_{i-1}) = \nu(\Theta_i) \leq \sum_{j=0}^{i} \delta_j - 1.$$

Hence $q^i \delta_0 - \theta(i) \leq \delta_i$, and combining with (2), we get $\delta_i = q^i \delta_0 - \theta(i)$.

It follows from this argument that $\Theta_i \in M_i(\Theta_l)$ and $\Theta_{i-1} \in M_{i-1}(\Theta_l)$, for all $l$ such that $i \leq l \leq \bar{m}$, and hence $\varepsilon_i^{(l)} = \varepsilon_i^{(i)}$. It follows by induction that $\delta_i = q^i \delta_0 - \theta(i)$ for $i = 1, 2, \ldots, \bar{m} - 1$.

We have

$$\varepsilon_{\bar{m}}^{(\bar{m})} = \nu(\Theta_{\bar{m}}) - \nu(\Theta_{\bar{m}-1}) = \sum_{i=0}^{\bar{m}} \delta_i - \left( \sum_{i=0}^{\bar{m}-1} \delta_i - 1 \right) = \delta_{\bar{m}} + 1,$$

and by Lemmas 3 and 4 and (10), we get

$$\delta_{\bar{m}} + 1 = \varepsilon_{\bar{m}}^{(\bar{m})} \leq q \varepsilon_{\bar{m}-1}^{(\bar{m})} \leq q \varepsilon_{\bar{m}-1}^{(\bar{m}-1)} = q^{\bar{m}} \delta_0 - \theta(\bar{m}) + 1.$$

Combining with the lower bound from (1) we get

$$\delta_{\bar{m}} = \varepsilon_{\bar{m}}^{(\bar{m})} - 1 = q^{\bar{m}} \delta_0 - \theta(\bar{m}),$$

and the theorem follows by induction. $\square$

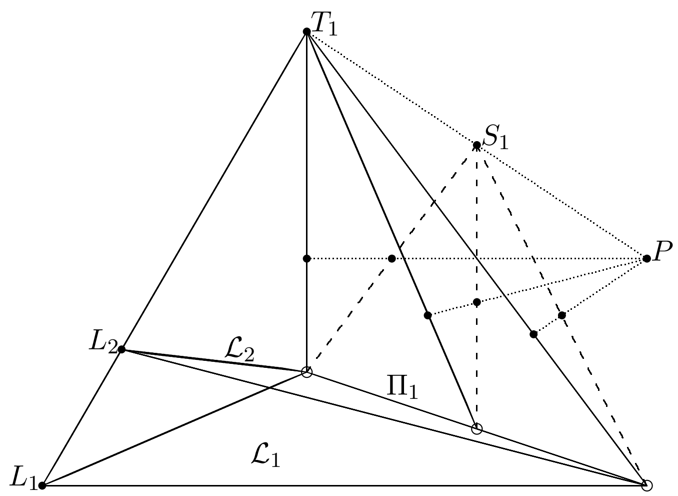


Figure 1. Representation of $\mathsf{PG}(4,2)$ for the proof of Theorem 9. Black lines are in $\Pi_3$, dashed lines in $\Pi_2$, and dotted lines are in neither. White points are in $\Pi_1$. The point $L_1$ and $\Pi_1$ span $\mathcal{L}_1$, and $L_2$ and $\Pi_1$ span $\mathcal{L}_2$.

**Corollary 7** *Let $\mathcal{C}$ be an $m$-optimal code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ for some $m$ such that $1 \leq m \leq k-1$. For every $\Pi_m \in M_m$, $\nu|_{\Pi_m}$ corresponds to a chained code with difference sequence $(\delta_0 - 1, \delta_1, \delta_2, \ldots, \delta_{m-1}, \delta_m + 1)$.*

**PROOF.** In the proof of Theorem 1 we proved that $\Theta_i \in M_i(\Theta_m) = M_i(\Pi_m)$, and we found the difference sequence as given in the corollary. $\square$

**Remark 8** *We know from Lemma 6 that $\delta_0 \geq 2$, so the difference sequence has only positive elements as expected. Writing*

$$(\varepsilon_0 = \delta_0 - 1, \varepsilon_1 = \delta_1, \ldots, \varepsilon_{m-1} = \delta_{m-1}, \varepsilon_m = \delta_m + 1)$$

*for the difference sequence of $\nu|_{\Pi_m}$, we have $\varepsilon_i = q\varepsilon_{i-1} - 1$ for $i = 1, \ldots, m-1$ and $\varepsilon_m = q\varepsilon_{m-1}$.*

### *2.2 Binary case*

For binary codes we have a special bound, which also implies that binary codes cannot be $(k-1)$-optimal if $k \geq 3$.

**Theorem 9** *If $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 3$, is a binary ENDS, then*

$$\delta_{k-1} \leq 2^{k-2}\delta_1 - 2 - 2^{k-2}.$$

**PROOF.** Take $\Pi_{k-1} \in M_{k-1}$ and $\Pi_{k-2} \in M_{k-2}$, and let

$$\Pi_{k-3} Pi_{k-1} \cap \Pi_{k-2}.$$

Because the code is extremal non-chain, $\Pi_{k-3}$ is a $(k-3)$-space. Also let $\{P\} \in M_0$.

Define

$$\begin{aligned}
\mathcal{S} &:= \Pi_{k-2} \backslash \Pi_{k-3} = \{S_i \mid i = 1, 2, \ldots, 2^{k-2}\}, \\
\ell_i &:= \langle P, S_i \rangle = \{P, S_i, T_i\}, \quad i = 1, 2, \ldots, 2^{k-2}.
\end{aligned}$$

Every line through $P$ meets $\Pi_{k-1}$, so the points $T_i$ are in $\Pi_{k-1}$. Define the set

$$\mathcal{T} := \{T_i \mid i = 1, 2, \ldots, 2^{k-2}\}.$$

Because the code is an ENDS, $\nu(\ell_i) \leq \delta_0 + \delta_1 - 1$ for all $i$. Hence

$$\begin{aligned}
&\nu(T_i) \leq \delta_1 - \nu(S_i) - 1, \quad i = 1, 2, \ldots, 2^{k-2}, \\
&\nu(\mathcal{T}) \leq 2^{k-2}\delta_1 - \nu(\mathcal{S}) - 2^{k-2}.
\end{aligned}$$

We know that

$$\nu(\Pi_{k-2}) = \nu(\mathcal{S}) + \nu(\Pi_{k-3}) = \sum_{i=0}^{k-2} \delta_i,$$

so

$$\nu(\mathcal{T}) - \nu(\Pi_{k-3}) \leq 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i.$$

The join of $\{P\}$ and $\Pi_{k-2}$ is a $(k-1)$-space, intersecting $\Pi_{k-1}$ in a $(k-2)$-space, namely $\mathcal{T} \cup \Pi_{k-3}$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the other two distinct $(k-2)$-spaces such that $\Pi_{k-3} \subset \mathcal{L}_i \subset \Pi_{k-1}$ for $i = 1, 2$.

Now we have

$$\begin{aligned}
\sum_{i=0}^{k-1} \delta_i = \nu(\Pi_{k-1}) &= \nu(\mathcal{L}_1) + \nu(\mathcal{L}_2) - \nu(\Pi_{k-3}) + \nu(\mathcal{T}) \\
&\leq 2\left(\sum_{i=0}^{k-2} \delta_i - 1\right) + 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i.
\end{aligned}$$

This is simplified to

$$\delta_{k-1} \leq 2^{k-2}\delta_1 - 2^{k-2} - 2,$$

and the theorem is proved. $\square$

**Theorem 10 (Total value)** *If $k \geq 2$, $1 \leq m \leq k-1$, and $(\delta_0, \delta_1, \ldots \delta_k)$ satisfies $(\mathrm{N}m-1.m)$, then*

$$\nu(\mathsf{PG}(k,q)) \leq \sum_{i=0}^{m-1} \delta_i + (\delta_m - 1) \sum_{i=0}^{k-m} q^i.$$

**PROOF.** Let $\alpha \in M_{m-1}$. In $\mathsf{PG}(k,q)$ there are $\theta(k-m)$ $m$-spaces containing $\alpha$, and for every such $m$-space $\beta \supset \alpha$, we know by condition $(\mathrm{N}m-1.m)$ that

$$\nu(\beta \backslash \alpha) \leq \delta_m - 1.$$

Thus $\nu(\mathsf{PG}(k,q) \backslash \alpha) \leq (\delta_m - 1)\theta(k-m)$. By the definition of $\alpha$, we know that

$$\nu(\alpha) = \sum_{i=0}^{m-1} \delta_i,$$

and the theorem follows. $\square$

For an ENDS, several bounds may be derived from the above theorem. Corollary 11 is the best possible bound for $(k-1)$-optimal codes, while Corollary 12 is stronger for binary codes.

**Corollary 11** *If $(\delta_0, \delta_1, \ldots \delta_k)$ is a difference sequence satisfying $(\mathrm{N}0.1)$ and $k \geq 2$, then*

$$\nu(\mathsf{PG}(k,q)) \leq \delta_0 + (\delta_1 - 1) \sum_{i=0}^{k-1} q^i \leq \sum_{i=0}^{k} q^i \delta_0 - (q+2) \sum_{i=0}^{k-1} q^i.$$

*The bound holds with equality if and only if every line through $X \in M_0$ has value $(q+1)\delta_0 - (q+2)$.*

**Corollary 12** *If $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 2$, satisfies $(\mathrm{N}k-2.k-1)$, then $\delta_k \leq q\delta_{k-1} - (q+1)$.*

**Theorem 13** *Let $2 \leq k \leq 4$. Then the given bounds on $\delta_1$ through $\delta_k$ are the best possible. In particular there exists a construction meeting the bounds with* *equality if and only if the following constraint on* $\delta_0$ *is met*

$$\begin{array}{llll} \delta_0 \geq 3 & \text{if} & q=2 & \wedge \quad k=2 \\ \delta_0 \geq 5 & \text{if} & q=2 & \wedge \quad k=3 \\ \delta_0 \geq 4 & \text{if} & q=2 & \wedge \quad k=4 \\ \delta_0 \geq 2 & \text{if} & q=3 & \wedge \quad k=2 \\ \delta_0 \geq 3 & \text{if} & q=3 & \wedge \quad k=3,4 \\ \delta_0 \geq 2 & \text{if} & q\geq 4 & \wedge \quad k=2,3 \\ \delta_0 \geq 3 & \text{if} & q\geq 4 & \wedge \quad k=4. \end{array}$$

The theorem has been proved by giving explicit constructions. Chen and Kløve proved it for $k = 3$ and $q \geq 3$ in [2] and for $k = 3$ and $q = 2$ in [5]. It was proved for $k = 4$ in [12]. The example below shows it for $k = 2$. For $k \leq 1$, there are no non-chain conditions.

**Example 14** *An optimal ENDS in* $\mathsf{PG}(2, q)$ *is easily obtained as follows. Let* $\ell$ *be a line, and* $X \notin \ell$ *a point. Let* $\nu(X) = \delta_0$*. Consider each line* $\alpha \ni X$*. If* $q \geq 3$*, we choose two points in* $\alpha \backslash (\{X\} \cup \ell)$ *to have value* $\delta_0 - 2$*. All remaining points have value* $\delta_0 - 1$*. Note that* $\delta_0 \geq 2$*.*

*If* $q = 2$*, there is only one point in* $\alpha \backslash (\{X\} \cup \ell)$*, so that point must have value* $\delta_0 - 3$*, thus* $\delta_0 \geq 3$*.*

## 3 Structure of optimal codes

In this section we will find further necessary conditions for an extremal non-chain code to be $m$-optimal. For instance if $\mathcal{H} \in M_3$ is a 3-space of maximum value, then there are a line $\ell \subseteq \mathcal{H}$ and a plane $\mathcal{P} \subseteq \mathcal{H}$ such that

$$\begin{array}{ll} \nu(p) = \delta_0 - 3 & \forall p \in \ell \cap \mathcal{P} \\ \nu(p) = \delta_0 - 2 & \forall p \in \ell \cup \mathcal{P}, \quad p \notin \ell \cap \mathcal{P} \\ \nu(p) = \delta_0 - 1 & \qquad \text{otherwise}. \end{array}$$

The general result is stated in Theorem 26.

**Lemma 15** *If* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k$*, then*

$$\sum_{i=m}^{k} \delta_i = \theta(k-m)\delta_m - \sum_{i=0}^{k-m-1} \theta(i), \quad 0 \leq m \leq k.$$

**PROOF.** The equality follows immediately from the fact that if $0 \le i \le m \le k$, then

$$\delta_m = q^i \delta_{m-i} - \theta(i-1).$$

□

**Lemma 16** *If* $0 \le a \le q-1$, *then*

$$\theta(m) - a \sum_{i=0}^{m-1} \theta(i) \ge 1.$$

**PROOF.** We write

$$\begin{aligned} \theta(m) - a \sum_{i=0}^{m-1} \theta(i) &= \theta(m) - \frac{a}{q-1} \sum_{i=0}^{m-1} (q^{i+1} - 1) \\ &= \theta(m) - \frac{a}{q-1} (\theta(m) - 1 - m) \ge 1. \end{aligned}$$

□

**Lemma 17** *Let* $\nu$ *be a value assignment with difference sequence* $(\delta_0, \delta_1, \ldots, \delta_k)$ *where* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k-1$. *If* $\Pi_m \in M_m$, *then* $\nu|_{\Pi_m}$ *has difference sequence* $(\delta_0, \delta_1, \ldots, \delta_m)$.

**PROOF.** The proof is trivial for $m = k$, so assume $m < k$. Let

$$\emptyset = \Theta_{-1} \subset \Theta_0 \subset \Theta_1 \subset \ldots \subset \Theta_m = \Pi_m$$

be a chain of subspaces such that $\Theta_i$ has the greatest value among the $i$-spaces containing $\Theta_{i-1}$ in $\Pi_m$. Define $\delta'_i = \nu(\Theta_i) - \nu(\Theta_{i-1})$.

Let $(\delta''_0, \delta''_1, \ldots, \delta''_m)$ be the difference sequence of $\nu|_{\Pi_m}$. It is sufficient to prove that $\delta'_i = \delta_i$ for all $i$, because

$$\sum_{i=0}^{j} \delta'_i \le \sum_{i=0}^{j} \delta''_i \le \sum_{i=0}^{j} \delta_i, \quad 0 \le j \le m. \tag{11}$$

Suppose for contradiction that there is an $i$ such that $\delta_i \ne \delta'_i$. Let $l$ be the smallest such $i$. Note that $\delta'_l < \delta_l$ by (11).

Since there are only $\theta(m-l)$ distinct $l$-spaces containing $\Theta_{l-1}$ in $\Pi_m$, we get

$$\nu(\Pi_m) \le \theta(m-l)\delta'_l + \sum_{i=0}^{l-1} \delta'_i \le \theta(m-l)(\delta_l - 1) + \sum_{i=0}^{l-1} \delta_i.$$

Also note that by Lemma 15,

$$\nu(\Pi_m) = \theta(m-l)\delta_l - \sum_{j=0}^{m-l-1} \theta(j) + \sum_{i=0}^{l-1} \delta_i.$$

Combine the two lines to get

$$\theta(m-l)\delta_l - \sum_{j=0}^{m-l-1} \theta(j) + \sum_{i=0}^{l-1} \delta_i \le \theta(m-l)(\delta_l - 1) + \sum_{i=0}^{l-1} \delta_i,$$

which is equivalent to

$$\theta(m-l) - \sum_{j=0}^{m-l-1} \theta(j) \le 0,$$

contradicting Lemma 16. $\square$

**Corollary 18** *Any code with difference sequence $(\delta_0, \delta_1, \ldots, \delta_k)$ such that $\delta_i = q\delta_{i-1} - 1$ for $i = 1, \ldots, k-1$ satisfies the chain condition.*

**Lemma 19** *Let $\nu$ be a value assignment with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ such that $\delta_k = q\delta_{k-1}$. For every $(k-1)$-space $\Pi_{k-1} \supset \Pi_{k-2} \in M_{k-2}$, we have $\Pi_{k-1} \in M_{k-1}$.*

**PROOF.** Consider $\Pi_{k-2} \in M_{k-2}$. Let $B_0, \ldots, B_q$ be the $(k-1)$-spaces such that $\Pi_{k-2} \subset B_j$, $j = 0, \ldots, q$. We get

$$\nu(\mathsf{PG}(k,q)) = \sum_{j=0}^{q} \nu(B_j \backslash \Pi_{k-2}) + \nu(\Pi_{k-2}) = \sum_{j=0}^{k} \delta_j.$$

Since $\delta_k = q\delta_{k-1}$, we get that

$$(q+1)\delta_{k-1} = \sum_{j=0}^{q} \nu(B_j \backslash \Pi_{k-2}).$$

Comparing this with the fact that $\nu(B_j \backslash \Pi_{k-2}) \le \delta_{k-1}$ for all $j$, we get that $B_j \in M_{k-1}$, as required. $\square$

We recall Corollary 7 and Remark 8 to get the following corollary.

**Corollary 20** *If $(\delta_0, \delta_1, \ldots \delta_k)$ is a 1-optimal ENDS, $k \ge 2$, and $\ell$ is line with value $\nu(\ell) = \delta_0 + \delta_1$, then $\nu(p) = \delta_0 - 1$ for all $p \in \ell$.*

**Lemma 21** *Let $\nu : \mathsf{PG}(k,q) \to \mathbb{N}_0$ be a value assignment with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ such that $\delta_i = q\delta_{i-1} - 1$, $1 \le i \le k$. For every $\Pi_{m-1} \in M_{m-1}$, $0 \le m \le k$, we have that*

*(a) the number of distinct $m$-spaces of maximum value through $\Pi_{m-1}$ is at least*

$$\theta(k-m) - \sum_{j=0}^{k-m-1} \theta(j).$$

*(b) for $m = k-1$ there is a unique $m$-space $\Pi_m \notin M_m$ such that $\Pi_{m-1} \subset \Pi_m$, and*

$$\nu(\Pi_m) = \sum_{j=0}^{m} \delta_j - 1.$$

**PROOF.** There are $\theta(k-m)$ $m$-spaces $B_i \supset \Pi_{m-1}$. We get that

$$\nu(\mathsf{PG}(k,q)) = \sum_{j=1}^{\theta(k-m)} \nu(B_j \backslash \Pi_{m-1}) + \nu(\Pi_{m-1}) = \sum_{j=0}^{k} \delta_j.$$

and by Lemma 15,

$$\sum_{j=1}^{\theta(k-m)} \nu(B_j \backslash \Pi_{m-1}) = \sum_{j=m}^{k} \delta_j = \theta(k-m)\delta_m - \sum_{j=0}^{k-m-1} \theta(j).$$

Clearly

$$\nu(B_j \backslash \Pi_{m-1}) \le \delta_m, \quad 1 \le j \le \theta(k-m). \tag{12}$$

Comparing the last two equations, we note that at least

$$\theta(k-m) - \sum_{j=0}^{k-m-1} \theta(j)$$

of the $B_i$ give equality in (12). If $m \le k-1$, then at least one of the $B_j$ gives inequality. The case where $m = k-1$, is just a special case where $q$ of the $B_i$ gives equality and one gives inequality. The exact value of the one with inequality is easily computed. □

**Lemma 22** *Let $C$ be a code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. If $\delta_i = q\delta_{i-1} - 1$ for $i = 1, \ldots, k$, then there exists at most one point which is not contained in any element of $M_{k-1}$.*

**PROOF.** Suppose there are two distinct points $P, Q \in \mathsf{PG}(k,q)$ which are not contained in any element of $M_{k-1}$. Consider a chain

$$\Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{k-1} \subset \mathsf{PG}(k,q),$$

such that $\Pi_i \in M_i$ for each $i = 0, \ldots, k-1$. Let $\ell$
$spP, Q$. Obviously there is a point $S \in \ell \cap \Pi_{k-1}$. By assumption $P, Q \notin \Pi_{k-1}$, so $S \neq P$ and $S \neq Q$.

We claim that we can assume that $S \notin \Pi_{k-2}$. By Lemma 21b, there are $q$ points in $\Pi_1$ which are elements of $M_0$, so if $S \in \Pi_0$, we can replace $\Pi_0$ by some other point which is in $\Pi_1$ and in $M_0$. For all $i$ such that $1 \leq i \leq k-2$, there are $q$ $i$-spaces in $M_i$ containing $\Pi_{i-1}$ in $\Pi_{i+1}$. Thus if $S \in \Pi_i \backslash \Pi_{i-1}$, we can replace $\Pi_i$ with some other $i$-space, maintaining the chain. By induction we can assume that $S \notin \Pi_{k-2}$, as required.

There are $q+1$ distinct $(k-1)$-spaces spanned by $\Pi_{k-2}$ and a point on $\ell$, and only one of these is not an element of $M_{k-1}$ by Lemma 21b. Since $\langle P \rangle \Pi_{k-2}$ and $\langle Q \rangle \Pi_{k-2}$ are two distinct $(k-1)$-spaces, either $P$ or $Q$ is contained in some element of $M_{k-1}$. The lemma follows by contradiction. $\square$

**Lemma 23** *Let $\nu$ be a value assignment with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ such that $k \leq 2$ and $\delta_i = q\delta_{i-1} - 1$ for $1 \leq i \leq k$. Then there exists a collection $S$ containing exactly one $i$-space for each $i = 0, \ldots, k-1$ such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(k,q).$$

**PROOF.** For $k = 0$ the result is trivial.

For $k = 1$ there are $q+1$ points. By Lemma 21b there is one point $P$ of value $\delta_0 - 1$ and $q$ points of value $\delta_0$. Hence $S = \{P\}$ forms the required collection.

Consider $k = 2$. There is a point $\wp \in M_0$. Let $\ell_0, \ldots, \ell_q$ be the distinct lines such that $\wp \subset \ell_i$ for all $i$. One of these lines, say $\ell_0$, has value $\delta_1 + \delta_0 - 1$, while the remaining $q$ lines have value $\delta_0 + \delta_1$ by Lemma 21b. This means that for $1 \leq i \leq q$, there is exactly one point $\alpha_i \in \ell_i$ such that $\nu(\alpha_i) = \delta_0 - 1$. There are at most two points in $\ell_0$ with value $\delta_0 - 1$ or less. The remaining points have value $\delta_0$. Obviously, every line in $\mathsf{PG}(2,q)$ has value at most $\delta_0 + \delta_1$, and hence has at least one point of value $\delta_0 - 1$ or less. A set of $q+2$ points cannot meet every line in a plane unless it contains a line [10, Lemma 13.4(iv)]. It follows that there must be a line $\Pi_1$ such that $\nu(p) \leq \delta_0 - 1$ for all $p \in \Pi_1$. Since $\nu(\ell_0) = \delta_1 + \delta_0 - 1$, there is either one point $\Pi_0 = \Pi_1 \cap \ell_0$ which has value $\delta_0 - 2$ or two distinct points $\Pi_0$ and $\Pi_1 \cap \ell_0$ of value $\delta_0 - 1$. In either case $\{\Pi_0, \Pi_1\}$ forms the required collection $S$. $\square$

**Definition 24 (Projections)** *We define the* projection $\pi_p$ *of* $\mathsf{PG}(k,q)$ *through the point* $p \in \mathsf{PG}(k,q)$*:*

$$\pi_p : \ \mathsf{PG}(k,q) \to \mathsf{PG}(k-1,q),$$

*by mapping distinct lines through* $p$ *in* $\mathsf{PG}(k,q)$ *to distinct points in* $\mathsf{PG}(k-1,q)$ *such that coplanar lines are taken to collinear points. We define the* projected value assignment

$$\begin{aligned} \nu_p &: \ \mathsf{PG}(k-1,q) \to \mathbb{N}_0, \\ \nu_p &: \ X \mapsto \nu(\pi_p^{-1}(X)\backslash\{p\}). \end{aligned}$$

The code corresponding to $\nu_p$ is the subcode $\langle p\rangle^*$ of codimension 1 [7].

**Lemma 25** *Let* $\nu : \ \mathsf{PG}(k,q) \to \mathbb{N}_0$, $q \geq 3$, *be the value assignment of a code* $\mathcal{C}$ *with difference sequence* $(\delta_0, \delta_1, \ldots \delta_k)$ *such that* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k$. *Then there exists a collection* $S$ *containing exactly one* $i$*-space for each* $i = 0, \ldots, k-1$ *such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(k,q).$$

**PROOF.** Lemma 23 proves it for $k < 3$. Now assume that the lemma holds for $k < n$, and consider

$$\nu : \ \mathsf{PG}(n,q) \to \mathbb{N}_0, \quad n \geq 3 \ \wedge \ q \geq 3.$$

For $\Pi_k \in M_k$, $k < n$, let $S(\Pi_k)$ be the collection $S$ corresponding $\nu|_{\Pi_k}$. By Lemma 17 $\nu|_{\Pi_k}$ has difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. Thus $S(\Pi_k)$ exists by the induction hypothesis, and it has the property given in the lemma. We define $\sigma_i(\Pi_k)$ to be the $i$-space in $S(\Pi_k)$.

**Claim I** *If* $\Theta_1 \in M_{n-2}$ *and* $\Theta_2 \in M_{n-1}$ *such that* $\Theta_1 \subset \Theta_2$, *then*

$$\sigma_i(\Theta_1) = \Theta_1 \cap \sigma_{i+1}(\Theta_2), \quad 0 \leq i \leq n-3.$$

We can use either $S(\Theta_1)$ or $S(\Theta_2)$ to express the value of a point $p \in \Theta_1$. Hence

$$\#\{\Pi \in S(\Theta_1) \mid p \in \Pi\} = \#\{\Pi \in S(\Theta_2) \mid p \in \Pi\}. \tag{13}$$

For all $i$, $\sigma_i' := \sigma_{i+1}(\Theta_2) \cap \Theta_1$ is either an $(i+1)$-space if $\sigma_{i+1}(\Theta_2) \subseteq \Theta_1$, or else an $i$-space. Equation (13) can only be satisfied for all $p \in \Theta_1$ if $\dim \sigma_i' = i$ for all $i$. Hence we can let $\sigma_i'$ for $i \geq 0$ be the elements of $S(\Theta_1)$, and the claim follows.

**Claim II** *If $1 \leq i \leq n-2$, then there is an $(i+1)$-space $\sigma_{i+1}$ such that $\sigma_i(\mathcal{A}) \subset \sigma_{i+1}$ for all $\mathcal{A} \in M_{n-1}$.*

Consider $P \in M_{n-3}$, $\alpha_0 \in M_{n-2}$, $\mathcal{A}_1, \dots, \mathcal{A}_q \in M_{n-1}$, and an $(n-1)$-space $\mathcal{A}_0 \notin M_{n-1}$ such that $P \subset \alpha_0 \subset \mathcal{A}_j$ for $0 \leq j \leq q$. Since $q \geq 3$, there are at least two distinct $(n-2)$-spaces $\alpha_1, \alpha_2 \in M_{n-2}$ such that $P \subset \alpha_j \subset \mathcal{A}_1$ and $\alpha_0 \neq \alpha_j$ for $j = 1, 2$. There are also at least two distinct $(n-2)$-spaces $\beta_1, \beta_2 \in M_{n-2}$ such that $P \subset \beta_j \subset \mathcal{A}_2$ and $\alpha_0 \neq \beta_j$ for $j = 1, 2$. Define $\sigma_{i+1}$ $sigma_i(\mathcal{A}_1)\sigma_i(\mathcal{A}_2)$. We have $\mathcal{A}_1 \cap \mathcal{A}_2 = \alpha_0 \in M_{n-2}$, so

$$\sigma_{i-1}(\alpha_0) = \sigma_i(\mathcal{A}_1) \cap \alpha_0 = \sigma_i(\mathcal{A}_2) \cap \alpha_0 = \sigma_i(\mathcal{A}_1) \cap \sigma_i(\mathcal{A}_2),$$

by Claim I. Since $\dim \sigma_{i-1}(\alpha_0) = i - 1$, we get $\dim \sigma_{i+1} = i + 1$. It remains to prove that $M_{n-1} = \mathfrak{S}$ where

$$\mathfrak{S} := \{\mathcal{A} \in M_{n-1} \mid \sigma_i(\mathcal{A}) \subset \sigma_{i+1}, 1 \leq i \leq n-2\}.$$

Consider the spaces $\alpha_1\beta_1$ and $\alpha_2\beta_1$. At least one of them is a space in $M_{n-1}$, denote it $\mathcal{B}_1$. Similarly, let $\mathcal{B}_2$ be either $\alpha_1\beta_2$ or $\alpha_2\beta_2$ such that $\mathcal{B}_2 \in M_{n-1}$. We have the following

$$\begin{aligned}
&\mathcal{B}_1 \cap \mathcal{A}_1 = \alpha_j \in M_{n-2}, \quad j = 1 \vee j = 2,\\
&\mathcal{B}_1 \cap \mathcal{A}_2 = \beta_1 \in M_{n-2},\\
&\mathcal{B}_2 \cap \mathcal{A}_1 = \alpha_j \in M_{n-2}, \quad j = 1 \vee j = 2,\\
&\mathcal{B}_2 \cap \mathcal{A}_2 = \beta_2 \in M_{n-2}.
\end{aligned}$$

It follows that $\sigma_i(\mathcal{B}_1) \cap \sigma_i(\mathcal{A}_1) = \sigma_{i-1}(\alpha_j)$ for $j = 1$ or $j = 2$, and $\sigma_i(\mathcal{B}_1) \cap \sigma_i(\mathcal{A}_2) = \sigma_{i-1}(\beta_1)$. Hence $\sigma_i(\mathcal{B}_1)$ meets $\sigma_{i+1}$ in two distinct $(i-1)$-spaces, and consequently $\sigma_i(\mathcal{B}_1) \subset \sigma_{i+1}$. A similar argument holds for $\mathcal{B}_2$, and hence $\sigma_i(\mathcal{B}_2) \subset \sigma_{i+1}$.

At least one of the $(n-2)$-spaces $\mathcal{A}_3 \cap \mathcal{B}_1$ or $\mathcal{A}_3 \cap \mathcal{B}_2$ is an element $\alpha' \in M_{n-2}$, because $P = \mathcal{A}_3 \cap \mathcal{B}_1 \cap \mathcal{B}_2 \in M_{n-3}$. It follows that $\sigma_i(\mathcal{A}_3)$ meets $\sigma_{i+1}$ in at least two distinct $(i-1)$-spaces, $\sigma_{i-1}(\alpha')$ and $\sigma_{i-1}(\alpha_0)$. We conclude that $\sigma_i(\mathcal{A}_3) \subset \sigma_{i+1}$. So far we have shown that

$$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{S}.$$

We note that if there are two distinct elements $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{S}$, and $\mathcal{A} \in M_{n-1}$ such that $\gamma_j$
$E_j \cap \mathcal{A} \in M_{n-2}$ for $j = 1, 2$ and $\gamma_1 \neq \gamma_2$, then $\sigma_i(\mathcal{A})$ meets $\sigma_{i+1}$ in two distinct $(i-1)$-spaces $\sigma_{i-1}(\gamma_j)$. Hence $\mathcal{A} \in \mathfrak{S}$.

If there are three distinct elements $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \in \mathfrak{S}$ and $\mathcal{A} \in M_{n-3}$ such that the intersections $\mathcal{E}_j \cap \mathcal{A}$ are three distinct $(n-2)$-spaces and

$$\mathcal{A} \cap \bigcap_{j=1}^{3} \mathcal{E}_j \in M_{n-3},$$

then at least two of the $\mathcal{E}_j$ meets $\mathcal{A}$ in distinct elements of $M_{n-2}$, and $\mathcal{A} \in \mathfrak{S}$.

Consider an element $\mathcal{A} \in M_{n-1}$ such that

$$P \subset \mathcal{A} \notin \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2\}.$$

If $\alpha_0 \not\subset \mathcal{A}$, then $\mathcal{A}$ meets $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ in three distinct $(n-2)$-spaces containing $P$, and thus $\mathcal{A} \in \mathfrak{S}$. If $\alpha_0 \subset \mathcal{A}$, then $\mathcal{A}$ meets $\mathcal{A}_1$, $\mathcal{B}_1$, and $\mathcal{B}_2$ in three distinct $(n-2)$-spaces containing $P$ and $\mathcal{A} \in \mathfrak{S}$. Thus we have proved that if $P \subset \mathcal{A} \in M_{n-1}$, then $\mathcal{A} \in \mathfrak{S}$.

If $\mathcal{A} \in M_{n-1}$ such that $\bar{P}\ P \cap \mathcal{A} \in M_{n-4}$, then there is $\xi \in M_{n-2}$ such that $P \subset \xi$ and $S := \xi \cap \mathcal{A} \in M_{n-3}$. This is obvious from the fact that there are at least $q^2 - 1$ $(n-2)$-spaces of maximum value through $P$ by Lemma 21, and at most $q + 2$ $(n-3)$-spaces through $\bar{P}$ in $\mathcal{A}$ that are not elements of $M_{n-3}$. Hence there are at least $q^2 - q - 3 \geq 3$ choices for $\xi$. There are at least three subspaces $\mathcal{E}_j \in M_{n-1}$, $j = 1, 2, 3$, through $\xi$, and

$$\mathcal{A} \cap \bigcap_{j=1}^{3} \mathcal{E}_j = S \in M_{n-3}.$$

Hence $\mathcal{A} \in \mathfrak{S}$.

Suppose for induction that if $P \not\subset \mathcal{A} \in M_{n-1}$ and there is $R \subseteq \bar{P}\ P \cap \mathcal{A}$ such that $R \in M_{j+1}$, then $\mathcal{A} \in \mathfrak{S}$. This was proved for $j = n-5$ in the last paragraph. It even holds when $n = 3$, because if $j = -2$, then $R = \emptyset \in M_{-1}$.

Consider $\mathcal{A} \in M_{n-1}$ such that there is $\bar{R} \in M_j$ such that $\bar{R} \subset \bar{P}$, but there is no $\bar{R}' \in M_{j+1}$ such that $\bar{R}' \subseteq \bar{P}$. Let $R \in M_{j+1}$ be such that $\bar{R} \subset R \subset P$. We shall prove that there is $\xi \in M_{n-2}$ such that $R \subset \xi$ and $\xi \cap \mathcal{A} \in M_{n-3}$. This is sufficient because then there are $q \geq 3$ elements of $\mathfrak{S}$ containing $\xi$ by the induction hypothesis, and at least two of them meet $\mathcal{A}$ in elements of $M_{n-2}$.

We prove the existence of $\xi$ by induction on $m$. Assume that

$$\exists R_m \in M_m, \text{ s.t. } R_m \cap \mathcal{A} \in M_{m-1}, \quad j+1 \leq m \leq n-3. \tag{14}$$

Let $R_{j+1} = R$. By Lemma 21, there are at least

$$\theta(n-(m+1)) - \sum_{l=0}^{n-(m+1)-1} \theta(l)$$

$(m+1)$-spaces of maximum value through $R_m$. Of these at most

$$\sum_{l=0}^{n-1-m-1} \theta(l)$$

meet $\mathcal{A}$ in an $m$-space which does not have maximum value. Hence at least

$$\theta(n-m-1) - 2\sum_{l=0}^{n-m-2} \theta(l) \geq 1$$

$(m+1)$-spaces satisfy (14) by Lemma 16. By induction $\xi$ $R_{n-2}$ exists, and hence $\mathfrak{S} = M_{n-1}$. This proves Claim II.

**Claim III** *For all $\mathcal{A} \in M_{n-1}$, $1 \leq i \leq n-2$, $\sigma_i(\mathcal{A}) = \sigma_{i+1} \cap \mathcal{A}$.*

By the previous claim it is sufficient to prove that $\sigma_{i+1} \not\subseteq \mathcal{A}$. Assume for contradiction that the claim fails for some $i$, and let $m$ be the largest such $i$. Let $\mathcal{A} \in M_{n-1}$ be such that $\sigma_{m+1} \subseteq \mathcal{A}$. Let $\mathcal{B} \in M_{n-1}$ such that $\sigma_m(\mathcal{A}) \neq \sigma_m(\mathcal{B})$. By Claim II we get that $\sigma_m(\mathcal{B}) \subset \sigma_{m+1} \subseteq \mathcal{A}$. Note that

$$\begin{aligned}
\#\sigma_m(\mathcal{B}) &= \theta(m) \\
\#(\sigma_m(\mathcal{A}) \cap \sigma_m(\mathcal{B})) &\leq \theta(m-1) \\
\# \bigcup_{j=0}^{m-1} \sigma_j(\mathcal{A}) &\leq \sum_{j=0}^{m-1} \theta(j).
\end{aligned}$$

Hence

$$\#\left(\sigma_m(\mathcal{B}) \setminus \bigcup_{i=0}^{m} \sigma_i(\mathcal{A})\right) \geq q^m - \sum_{j=0}^{m-1} \theta(j) \geq 1,$$

since $q \geq 3$. It follows that there exists

$$p \in \sigma_m(\mathcal{B}) \setminus \bigcup_{i=0}^{m} \sigma_i(\mathcal{A}).$$

Since the claim is assumed to hold for $i > m$, we have that

$$\begin{aligned}
\nu(p) &= \delta_0 - \#\{i \mid p \in \sigma_i(\mathcal{B}) \ \wedge\ 0 \leq i \leq n-2\} \\
&\leq \delta_0 - 1 - \#\{i \mid p \in \sigma_{i+1} \ \wedge\ m+1 \leq i \leq n-2\} \\
\nu(p) &= \delta_0 - \#\{i \mid p \in \sigma_i(\mathcal{A}) \ \wedge\ 0 \leq i \leq n-2\} \\
&= \delta_0 - \#\{i \mid p \in \sigma_{i+1} \ \wedge\ m+1 \leq i \leq n-2\},
\end{aligned}$$

and these two equations contradict each other, proving Claim III.

We write

$$U := \{\sigma_0(\mathcal{A}) \mid \mathcal{A} \in M_{n-1}\}.$$

Lemma 22 says that at most one point is not contained in any element of $M_{n-1}$. This means that we can form the set

$$S' = U \cup \{\sigma_i \mid i = 2, \ldots, n-1\},$$

giving the value of all points but at most one by the formula

$$\nu(p) = \delta_0 - \#\{\Pi \in S' \mid p \in \Pi\}.$$

**Claim IV** *There is a line $\sigma_1$ such that $\sigma_0(\mathcal{A}) \subset \sigma_1$ for all $\mathcal{A} \in M_{n-1}$.*

Take a point $\{F\} \in M_0$ such that

$$F \in \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{n-3} = P$$

is a chain of subspaces of maximum value. The projected value assignment $\nu_F$ defines an $(n-1)$-dimensional subcode code with weight $d_{n-1}$. The difference sequence of $\nu_F$ is $(\delta_1, \ldots, \delta_n)$, because $\pi_F(\Pi_i) \in M_{i-1}(\nu_F)$ for $0 \leq i \leq n$. By the induction hypothesis, there is a collection $S(\mathsf{PG}(n-1,q))$ of $i$-spaces $\sigma_i(\mathsf{PG}(n-1,q))$ for $i = 0, \ldots, n-2$ such that

$$\nu_F(p) = \delta_1 - \#\{\Pi \in S(\mathsf{PG}(n-1,q)) \mid p \in \Pi\}.$$

Clearly $F \notin \Pi$ for any $\Pi \in S'$. Hence $\pi_F(\sigma_i)$ is an $i$-space. We get the following formula for the values of every point but at most one in $\mathsf{PG}(n-1,q)$:

$$\begin{aligned} \nu_F(p) &= q\delta_0 - \#\{\Pi \in S' \mid p \in \pi_F(\Pi)\} \\ &= \delta_1 - \#\{\Pi \in S' \backslash \{\sigma_{n-1}\} \mid p \in \pi_F(\Pi)\}. \end{aligned}$$

It follows that

$$\begin{aligned} &\pi_F(\sigma_i) = \sigma_i(\mathsf{PG}(n-1,q)), \quad 2 \leq i \leq n-2 \\ &\pi_F(U) \subseteq \sigma_1(\mathsf{PG}(n-1,q)) \cup \sigma_0(\mathsf{PG}(n-1,q)). \end{aligned}$$

We have $U \cap \alpha_0 = \emptyset$ by Claim I. It follows that $\sigma_0(\mathcal{A}_i)$ for $i = 1, \ldots, q$ are $q$ distinct elements of $U$. Let $U'$ $U \backslash \mathcal{A}_0$ be the set of these $q$ points.

Now consider $V = \sigma_1(\mathsf{PG}(n-1,q)) \cup \sigma_0(\mathsf{PG}(n-1,q))$, the inverse image of which must consist of points in $U$ and points not contained in any element of $M_{n-1}$. In fact $\pi_F(U') \subset \sigma_1(\mathsf{PG}(n-1,q))$. Hence $U'$ are coplanar points.

There are more chains

$$F \neq F' \in \Pi'_0 \subset \Pi'_1 \subset \ldots \subset \Pi'_{n-3} \subset \alpha_0$$

of subspaces of maximum value. By projecting through such a point $F'$, we can show that $U'$ is also contained in a plane which is not equal to the first. Hence $U'$ is contained in a line, which we denote $\sigma_1$, and $\pi_F(\sigma_1) = \sigma_1(\mathsf{PG}(n-1,q))$

We shall prove that $U \cap \mathcal{A}_0 \subset \sigma_1$, and consequently that $U \subseteq \sigma_1$. This is trivial if $U \cap \mathcal{A}_0 = \emptyset$. Otherwise consider an arbitrary point $R \in U \cap A_0$. By the definition of $U$, there is $\mathcal{G} \in M_{n-1}$ such that $R \in \mathcal{G}$. By Lemma 17 there is a subspace $\rho \subset \mathcal{G}$ such that $\rho \in M_{n-2}$. By the argument used to prove Lemma 22, we can choose $\rho$ such that $R \notin \rho$. Projecting through a couple of distinct points contained in $M_0$ and in $\rho$, as we did in the previous paragraph, will show that $R \in \sigma_1$, as required. This proves Claim IV.

**Claim V** *There is a point $\sigma_0$ which is not contained in any element of $M_{n-1}$, and $S := \{\sigma_i \mid i = 0, \ldots, n-1\}$ forms the required collection such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \Pi_n. \tag{15}$$

First assume that $\sigma_0$ does exist. We have proved that (15) holds for all points except possibly for $\sigma_0$. If it does fail for $\sigma_0$, it must give us a wrong value for $\nu(\mathsf{PG}(n,q))$, but

$$\nu(\mathsf{PG}(n,q)) = \theta(n)\delta_0 - \sum_{\Pi \in S} \#\Pi = \theta(n)\delta_0 - \sum_{i=0}^{n-1} \theta(i) = \sum_{i=0}^{n} \delta_i,$$

by Lemma 15, and that is correct. If $\sigma_0$ did not exist, we would have no point in $S$, and the total value would not be correct. This completes the proof of Claim V and the lemma. $\square$

**Theorem 26** *Let $\mathcal{C}$ be a chained, non-binary code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. If*

$$\begin{aligned} \delta_i &= q\delta_{i-1} - 1, \quad i = 1, \ldots, k-1, \\ \delta_k &= q\delta_{k-1}, \end{aligned}$$

*then there exists a collection $S$ of exactly one $i$-space in $\mathsf{PG}(k,q)$ for each $i = 1, \ldots, k-1$, such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(k,q).$$

**PROOF.** Lemma 25 says that for each $\Pi_{k-1} \in M_{k-1}$, there is a set $S(\Pi_{k-1})$ such that

$$\nu(p) = \delta_0 - \#\{\Pi \in S(\Pi_{k-1}) \mid p \in \Pi\}, \quad \forall p \in \Pi_{k-1}.$$

Let $\sigma_i$ denote the $i$-space in $S$. If $k \geq 3$ we use the same argument as in the proof of Lemma 25, to show that

$$\sigma_i = \bigcup_{\Pi \in M_{k-1}} \sigma_{i-1}(\Pi), \quad i = 1, 2, \ldots, k-1.$$

Because every point is contained in some $\Pi_{k-1} \in M_{k-1}$, there is no point in $S$.

The cases for $k \leq 2$ are just as simple as the proof of Lemma 23. $\square$

This theorem will of course apply to every subspace $\Pi_m \in M_m(\mathcal{C})$ for an $m$-optimal, extremal non-chain code $\mathcal{C}$, and this fact has been most useful to limit the search for $m$-optimal constructions

**Corollary 27** *If $(\delta_0, \delta_1, \ldots \delta_k)$ is a 3-optimal ENDS where $k \geq 4$ and $q \geq 3$, then $\delta_0 \geq 3$.*

**PROOF.** Let $\Pi_3 \in M_3$, and apply the theorem on $\nu|_{\Pi_3}$. There is $p \in \Pi_3$, such that $\nu(p) = (\delta_0 - 1) - 2$. $\square$

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## References

[1] A. Beutelspacher. Blocking sets and partial spreads in finite projective spaces. *Geometrica Dedicata*, 9:425–449, 1980.

[2] Wende Chen and Torleiv Kløve. Bounds on the weight hierarchies of extremal non-chain codes of dimension 4. *Applicable Algebra in Engineering, Communication and Computing*, 8:379–386, 1997.

[3] Wende Chen and Torleiv Kløve. Bounds on the weight hierarchies of linear codes of dimension 4. *IEEE Trans. Inform. Theory*, 43(6):2047–2054, 1997.

[4] Wende Chen and Torleiv Kløve. Classification of the weight hierarchies of binary linear codes of dimension 4. Technical Report 147, Department of Informatics, University of Bergen, March 1998.

[5] Wende Chen and Torleiv Kløve. Weight hierarchies of extremal non-chain binary codes of dimension 4. *IEEE Trans. Inform. Theory*, 45(1):276–289, 1999.

[6] G. D. Cohen, S. B. Encheva, and G. Zémor. Antichain codes. *Proc. IEEE Intern. Symp. Inform. Theory*, page 232, August 1998. Full paper to appear in *Designs, Codes and Cryptography*.

[7] S. Dodunekov and J. Simonis. Codes and projective multisets. *Electron. J. Combin.*, 5(1), 1998. Research Paper 37.

[8] T. Helleseth, T. Kløve, and J. Mykkeltveit. The weight distribution of irreducible cyclic codes with block lengths $n_1((q^l-1)/n)$. *Discrete Mathematics*, 18:179–211, 1977.

[9] Tor Helleseth, Torleiv Kløve, and Øyvind Ytrehus. Generalized Hamming weights of linear codes. *IEEE Trans. Inform. Theory*, 38(3):1133–1140, 1992.

[10] J. W. P. Hirschfeld. *Projective Geometries over Finite Fields*. Oxford University Press, second edition, 1998.

[11] L. H. Ozarow and A. D. Wyner. Wire-tap channel II. *AT&T Bell Laboratories Technical Journal*, 63(10):2135–2157, December 1984.

[12] Hans Georg Schaathun. Upper bounds on weight hierarchies of extremal non-chain codes. Technical Report 171, Department of Informatics, University of Bergen, 1999. Also available at http://www.ii.uib.no/~georg/sci/inf/coding/public/.

[13] Michael A. Tsfasman and Serge G. Vlăduţ. Geometric approach to higher weights. *IEEE Trans. Inform. Theory*, 41(6, part 1):1564–1588, 1995.

[14] Victor K. Wei. Generalized Hamming weights for linear codes. *IEEE Trans. Inform. Theory*, 37(5):1412–1418, 1991.

[15] Victor K. Wei and Kyeongcheol Yang. On the generalized Hamming weights of product codes. *IEEE Trans. Inform. Theory*, 39(5):1709–1713, 1993.