The Binomial Theorem

Other uses of the binomial coefficient

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A common formula

- What is ...
  - $(x+y)^2 =$
  - $(x+y)^3 =$
  - $(x+y)^4 =$

    $\vdots$
  - $(x+y)^n =$
- The term $(x+y)$ is called a binomial

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The binomial theorem

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}xy^n.$$

Theorem (The Binomial Theorem)

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

... but why is it true?

Evolving a pattern

$$
\begin{aligned}
(x+y)^2 &= (x+y)(x+y) \\
&= (x+y)x + (x+y)y \\
&= xx + xy + xy + yy
\end{aligned}
$$

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Evolving a pattern

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= (x+y)x + (x+y)y\\ &= xx + xy + xy + yy\\ (x+y)^3 &= (x+y)^2(x+y) = (xx+xy+yx+yy)(x+y)\\ &= (xx+xy+yx+yy)x + (xx+xy+yx+yy)y\\ &= xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy\end{aligned}$$

Generalising

$$(x+y)^3 = xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy$$

- $(x+y)^n$ gives 2^n terms
  - (Ordered) lists of n elements from $\{x, y\}$
- Each of the 2^n possible lists occurs exactly once
- n factors: $(x+y)(x+y)\ldots(x+y)$
  - Each term of the sum will take one term from each factor

Generalising

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Simplifying

$$(x+y)^3 = xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy$$

- The order of the factors is irrelevant
  - e.g. $xxy = xyx = yxx$
- Two terms are equivalent (even equal) if the number of xs and ys are the same
- Consider $x^k y^{n-k}$
  - How many times does $x^k y^{n-k}$ occur in the sum?
  - How many ways can you choose k places for xs in an n-element list?
- As always; $\binom{n}{k}$

$$(x+y)^3 = x^3 + \binom{3}{1} x^2 y + \binom{3}{1} xy^2 + y^2$$

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$$(x+y)^3 = x^3 + \binom{3}{1} x^2 y + \binom{3}{1} xy^2 + y^2$$

Summary

Theorem (The Binomial Theorem)

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

- Each term is a obtained by selecting
    - either x or y
    - from each of the n factors $(x+y)$
- This gives initially *ordered* lists of variables
- The order is insignificant
    - $\binom{n}{k}$ terms are equal equal

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Exercise

Explain why

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$