Equivalence relations

Properties of relations

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Recollection

- Recall k-element permutations on a set S
- An ordered list of k elements from S

Equivalence relations

Two k-element permutations $p_1, p_2 \in P(S)$ are (set) equivalent $p_1 \sim p_2$ if they correspond to the same subset $S_k \subset S$.

- $\sim$ is a relation

$$R = \{(p_1, p_2) \mid p_1, p_2 \in P_S, p_1 \sim p_2\}$$

- $\sim$ partitions the set $P(S)$ into disjoint classes, each of size $k!$.
  - this allowed us to use the product principle
- p_1 and p_2 are in the same class if and only if $p_1 \sim p_2$

What properties must a relation have to partition the set into disjoint classes?

Reflexive relations

Note that $p_1 \sim p_1$ for any k-element permutation. We say that $\sim$ is reflexive.

Definition

A relation R on X is reflexive if xRx for any $x \in X$.

- $=$ is reflexive: $x = x$
- $\leq$ is reflexive: $x \leq x$
- $\subset$ is reflexive: $x \subset x$
- $<$ is not reflexive: $x \not< x$

Symmetric relations

If p_1 and p_2 correspond to the same set, tean p_2 and p_1 correspond to the same set.

I.e., if $p_1 \sim p_2$ then $p_2 \sim p_1$, and we say that $\sim$ is symmetric.

Definition

A relation R is symmetric if xRy whenever yRx.

- $=$ is symmetric
- $<, \leq, \subset$ are not symmetric
- *'is a neighbour of'* is symmetric (if A is next to B, then B is next to A)

Transitive relations

If $p_1 \sim p_2$ and $p_2 \sim p_3$, can we say anything about p_1 in relation to p_3?

- Each p_i correspond to a unique set S_i
- $p_1 \sim p_2$, so $S_2 = S_1$
- $p_2 \sim p_3$, so $S_3 = S_2 = S_1$
- Thus $p_1 \sim p_3$
- We say that $\sim$ is transitive

Definition

A relation R is transitive if xRy and yRz implies that xRz.

Equivalence relations

Definition

Definition (Equivalence Relation)

A relation $\sim$ which is reflexive, symmetric, and transitive is called an equivalence relation.

- Any equivalence relation defines a *partitioning*
  - into equivalence classes
- Let $\sim$ be an equivalence relation on a set S
  - for $x \in S$ define $[x] = \{y | y \in S, x \sim y\}$
  - $[x]$ is the equivalence class of x
- Each class contains only related elements

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Partitioning by equivalence

Theorem

Consider two equivalence classes, for some equivalence relation $\sim$ on a set S:

$$[x] = \{z | z \sim x\}$$
$$[y] = \{z | z \sim y\}$$

Either $[x] = [y]$ or $[x]$ and $[y]$ are disjoint.

Theorem

The union of all the equivalence classes $\{[x] | x \in S\}$ is S.

Thus the equivalence classes form a partioning of S.

Proof of the second theorem

- By reflexivity, $x \in [x]$
- Hence, for any $x \in S$, $x \in [x] \in \cup\{[x] \mid x \in S\}$
- and the union contains all elements of S

Proof of the first theorem

- Suppose there is $z \in [x] \cap [y]$
- Then $z \sim x$ and $z \sim y$
  - and $x \sim z$ and $y \sim z$ by symmetry
  - and $x \sim y$ and $y \sim x$ by transitivity
- Hence $x \in [y]$ and $y \in [x]$
- Consider any $w \in [y]$:
  - Because $x \in [y]$, $x \sim y \sim w$, and $w \in [x]$ by transitivity
  - Thus we conclude that $[y] \subset [x]$
- By a similar argument $[x] \subset [y]$
  - Hence $[x] = [y]$

Conclusion

Any equivalence relation forms a partitioning.

- Often (not always) the equivalence classes have equal size
- In those cases the product principle applies

Exercise

Which of the following relations are equivalence relations?

1. *'Is a brother of' on the set of people*
2. *'Is a sibling of' on the set of people*
3. *'Is a sister of' on the set of women*
4. *'Is a neighbour of' on the set of people living in a certain street*
5. *'Is a neighbour of' on the set of natural numbers (x and y are neighbours if $x - y = \pm 1$).*

Justify each answer.